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On a lemma of Scarf

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Abstract

The aim of this note is to point out some combinatorial applications of a lemma of Scarf, proved first in the context of game theory. The usefulness of the lemma in combinatorics has already been demonstrated in a paper by the first author and R. Holzman (J. Combin. Theory Ser. B 73 (1) (1998) 1) where it was used to prove the existence of fractional kernels in digraphs not containing cyclic triangles. We indicate some links of the lemma to other combinatorial results, both in terms of its statement (being a relative of the Gale–Shapley theorem) and its proof (in which respect it is a kin of Sperner's lemma). We use the lemma to prove a fractional version of the Gale–Shapley theorem for hypergraphs, which in turn directly implies an extension of this theorem to general (not necessarily bipartite) graphs due to Tan (J. Algorithms 12 (1) (1991) 154). We also prove the following result, related to a theorem of Sands et al. (J. Combin. Theory Ser. B 33 (3) (1982) 271): given a family of partial orders on the same ground set, there exists a system of weights on the vertices, which is (fractionally) independent in all orders, and each vertex is dominated by them in one of the orders.

1. Introduction

A famous theorem of Gale and Shapley [5] states that given a bipartite graph and, for each vertex v, a linear order $\leqslant_v$ on the set of edges incident with v, there exists a

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stable matching. Here, a matching M is called *stable* if for every edge $e \notin M$ there exists an edge in M meeting e and beating it in the linear order of the vertex at which they are incident. (The origin of the name "stable" is that in such a matching no non-matching edge poses a reason for breaking marriages: for every non-matching edge, at least one of its endpoints prefers its present spouse to the potential spouse provided by the edge.) Alternatively, a stable matching is a kernel in the line graph of the bipartite graph, where the edge connecting two vertices (edges of the original graph) is directed from the larger to the smaller, in the order of the vertex of the original graph at which they meet.

It is well known that the theorem fails for general graphs, as shown by the following simple example: let G be an undirected triangle on the vertices u, v, w, and define: $(u,v) >_u (w,u), (v,w) >_v (u,v), (w,u) >_w (v,w)$. But the theorem is true for general graphs if one allows fractional matchings, as follows easily from a result of Tan [12] (see Theorem 2.2). For example, in the example of the triangle one could take the fractional matching assigning each edge the weight $\frac{1}{2}$: each edge is then dominated at some vertex by edges whose sum of weights is 1 (for example, the edge (u,v) is dominated in this way at v).

The notions of stable matchings and fractional stable matchings can be extended to hypergraphs. A *hypergraphic preference system* is a pair $(H, \mathcal{O})$, where $H = (V, E)$ is a hypergraph, and $\mathcal{O} = \{\leqslant_v : v \in V\}$ is a family of linear orders, $\leqslant_v$ being an order on the set $D(v)$ of edges containing the vertex v. If H is a graph we call the system a *graphic* preference system.

A set M of edges is called a *stable matching* with respect to the preference system if it is a matching (that is, its edges are disjoint) and for every edge e there exists a vertex $v \in e$ and an edge $m \in M$ containing v such that $e \leqslant_v m$.

Recall that a function w assigning non-negative weights to edges in H is called a *fractional matching* if $\sum_{v \in h} w(h) \leqslant 1$ for every vertex v. A fractional matching w is called *stable* if every edge e contains a vertex v such that $\sum_{v \in h, e \leqslant_v h} w(h) = 1$.

As noted, by a result of Tan every graphic preference system has a fractional stable matching. Does this hold also for general hypergraphs? The answer is yes, and it follows quite easily from a result of Scarf [11]. This result is the starting point of the present paper. It was originally used in the proof of a better known theorem in game theory, and hence gained the name "lemma". Its importance in combinatorics has already been demonstrated in [2], where it was used to prove the existence of a fractional kernel in any digraph not containing a cyclic triangle.

Scarf's lemma is intriguing in that it seems unrelated to any other body of knowledge in combinatorics. The aim of this note is to link the lemma to other parts of combinatorics, both in terms of the family of results it belongs to, and of its proof. We shall show that the lemma is related to the Gale–Shapley theorem, and indicate similarity of its proof to that of Sperner's lemma (or, equivalently, to that of Brouwer's fixed point theorem).

In [4] it was noted that the Gale–Shapley theorem is a special case of a result of Sands et al. [10] on monochromatic paths in edge two-coloured digraphs. This result can also be formulated in terms of dominating antichains in two partial orders (see

Theorem 2.3). We shall use Scarf's lemma to prove a fractional generalisation of this theorem to an arbitrary number of partial orders.

In [4] a matroidal version of the Gale–Shapley theorem was proved, for two matroids on the same ground set. Using Scarf's lemma, we prove a fractional version of this result for arbitrarily many matroids on the same ground set.

We finish the introduction with a statement of the lemma which is duly called here "theorem". Apart from the original paper, a proof of it can also be found in [2]. The basic ideas of the proof are mentioned in the last section of the present paper.

Theorem 1.1 (Scarf [11]). *Let $n<m$ be positive integers, b a vector in $\mathbb{R}^n_+$. Also let $B=(b_{i,j}), C=(c_{i,j})$ be matrices of dimensions $n \times m$, satisfying the following three properties: the first n columns of B form an $n \times n$ identity matrix (i.e. $b_{i,j}=\delta_{i,j}$ for $i,j \in [n]$), the set $\{x \in \mathbb{R}^n_+ : Bx=b\}$ is bounded, and $c_{i,i}<c_{i,k}<c_{i,j}$ for any $i \in [n], i \neq j \in [n]$ and $k \in [m] \setminus [n]$.*

Then there is a nonnegative vector x of $\mathbb{R}^m_+$ such that $Bx=b$ and the columns of C that correspond to $\operatorname{supp}(x)$ form a dominating set, that is, for any column $i \in [m]$ there is a row $k \in [n]$ of C such that $c_{k,i} \leqslant c_{k,j}$ for any $j \in \operatorname{supp}(x)$.

2. Some applications

Let us start with a statement and proof of the generalisation of the Gale–Shapley theorem to hypergraphs as mentioned in the introduction.

Theorem 2.1. *In any hypergraphic preference system there exists a fractional stable matching.*

Proof. Let $(H, \mathcal{O})$ be a hypergraphic preference system, where $H=(V,E)$ and $\mathcal{O}=\{\leqslant_v : v \in V\}$. Let B be the incidence matrix of H, with the identity matrix adjoined to it at its left. Let C' be a $V \times E$ matrix satisfying the following two conditions:

(1) $c'_{v,e}<c'_{v,f}$ whenever $v \in e \cap f$ and $e <_v f$;
(2) $c'_{v,f}<c'_{v,e}$ whenever $v \in f \setminus e$.

Let C be obtained from C' by adjoining to it on its left a matrix so that C satisfies the conditions of Theorem 1.1. Let x be a vector as in Theorem 1.1 for B and C, where b is taken as the all 1's vector 1. Define $x'=x|_E$, namely the restriction of x to E. Clearly, x' is a fractional matching. To see that it is dominating, let e be an edge of H. By the conditions on x, there exists a vertex v such that $c_{v,e} \leqslant c_{v,j}$ for all $j \in \operatorname{supp}(x)$. Since $c_{v,v}<c_{v,e}$ it follows that $v \notin \operatorname{supp}(x)$. Since $Bx=\mathbf{1}$ it follows that $\operatorname{supp}(x)$ contains an edge f containing v (otherwise $(Bx)_v=0$). Since $c_{v,f} \geqslant c_{v,e}$ it follows by condition (2) above that $v \in e$. The condition $(Bx)_v=1$ now implies that e is dominated by x at v. $\square$

In fact, the vector x' can be assumed to be a vertex of the fractional matching polytope of H. To see this, write $x' = \sum \alpha_i y_i$, where $\alpha_i > 0$ for all $i, \sum \alpha_i = 1$ and the y_i's are vertices of the fractional matching polytope. Then each y_i must be a fractional stable matching. It is well known (see [8]) that the vertices of the fractional matching polytope of a graph are half integral, that is, they have only $0, \frac{1}{2}, 1$ coordinates. This yields the following result:

Theorem 2.2 (Tan [12]). *In any graphic preference system there exists a half-integral fractional stable matching. In other words, there exists a set of edges S whose connected components are single edges and cycles, such that every edge e of the graph contains a vertex $v \in \cup S$ such that $e \leqslant_v s$ for each $s \in S$ containing v.*

Tan's original proof is based on a polynomial algorithm of Irving [6] for testing the existence of an (integral) stable matching in a graphic preference system. Thus his paper contains a polynomial time algorithm for finding stable fractional matchings in graphs. He also proved the following interesting fact: if an odd cycle appears in the support of a fractional stable matching, then it appears in the support of all stable fractional matchings. This means that the existence of a stable matching is equivalent to the non-existence of an odd cycle in the support of any given fractional matching. In [1], this latter result of Tan is proved independently from Irving's algorithm. Also in [1], there is given a direct reduction of Theorem 2.2 to 2.1 (that is, not using the half-integrality of vertices of the matching polytope).

In [10], a generalisation of the Gale–Shapley theorem by Sands et al. was proved. Its original formulation was in terms of paths in digraphs whose edges are two-coloured. However, at its core lies the following fact about pairs of partial orders.

Let V be a finite ground set and $\leqslant_1$ and $\leqslant_2$ be two partial orders on V. A *dominating common antichain of $\leqslant_1$ and $\leqslant_2$* is a subset A of V such that A is an antichain in both partial orders and for any element v of V there is an element a in A with $v \leqslant_1 a$ or $v \leqslant_2 a$.

Theorem 2.3 (See Fleiner [3,4]). *For any two partial orders $\leqslant_1$ and $\leqslant_2$ on the same finite ground set V, there exists a dominating antichain of $\leqslant_1$ and $\leqslant_2$.*

The Gale–Shapley theorem is obtained by applying this theorem to the two orders on the edge set of the bipartite graph, each being obtained by taking the (disjoint) union of the linear orders induced by the vertices in one side of the graph.

The theorem is false for more than two partial orders. But a fractional version is true. For given partial orders $\leqslant_1, \leqslant_2, \dots, \leqslant_k$ on a ground set V, a nonnegative vector x of $\mathbb{R}_+^V$ is called a *fractional dominating antichain* if x is a *fractional antichain* (i.e. $\sum_{c \in C} x(c) \leqslant 1$ for any chain C of any of the partial orders $\leqslant_i$) and x is a *fractional upper bound* for any element of V, that is for each element v of V there is a chain $v = v_0 \leqslant_i v_1 \leqslant_i v_2 \leqslant_i \cdots \leqslant_i v_l$ of some partial order $\leqslant_i$ with $\sum_{j=0}^{l} x(v_j) = 1$. Note that if a fractional dominating antichain x happens to be integral then it is the characteristic vector of a dominating antichain.

Theorem 2.4. *Any finite set $\leqslant_1, \leqslant_2, \dots, \leqslant_k$ of partial orders on the same finite ground set V has a fractional dominating antichain.*

Proof. For each $i \leqslant k$ let $\mathscr{D}_i$ be the set of maximal chains in the partial order $\leqslant_i$. Let $\mathscr{J} = \bigcup_{i \leqslant k} \{i\} \times \mathscr{D}_i$ (that is, $\mathscr{J}$ is the union, with repetition, of the families $\mathscr{D}_i$).

Let B' be the $\mathscr{J} \times V$ incidence matrix of the chains of $\mathscr{J}$ (that is, for $v \in V$ and a maximal chain D in $\leqslant_i$, the $((i,D),v)$ entry of B' is 1 if $v \in D$, otherwise it is 0). Let $B := [I_n, B']$ be obtained by adding an $n \times n$ identity matrix I_n in front of B'.

Next we define a $\mathscr{J} \times V$ matrix C'. For $v \in V$ and $j = (i,D) \in \mathscr{J}$ define $C'_{j,v}$ as $|D| + 1$ if $v \notin D$, and as the height of v in D in the order $\leqslant_i$ if $v \in D$. Append now on the left of C' a matrix so that the resulting matrix C satisfies the conditions of Theorem 1.1.

Applying Theorem 1.1 to the above matrices B, C and the all 1's vector $b = \mathbf{1}$, we get a nonnegative vector $x \in \mathbb{R}^{\mathscr{J} \cup V}$. Let x' be the restriction of x to $\mathbb{R}^V$. As $B \cdot x = b = \mathbf{1}$, we have $B' \cdot x' \leqslant \mathbf{1}$, meaning that x' is a fractional antichain. The domination property of x implies that for any element v of V there is a chain D of some partial order $\leqslant_i$ such that for any element u from $D \cap \operatorname{supp}(x)$ we have $v \leqslant_i u$. Since $c_{(i,D),(i,D)}$ is smallest in row (i,D) of C, it follows that the column (i,D) of C does not belong to $\operatorname{supp}(x)$. The equality $(Bx)_{(i,D)} = 1$ thus means that $\sum_{d \in D} x(d) = 1$, showing that $\sum_{d \in D, d \geqslant_i v} x(d) = 1$. This proves the fractional upper bound property of x. $\square$

Our last application is a generalisation of a matroid version of the Gale–Shapley theorem.

An *ordered matroid* is a triple $\mathscr{M} = (E, \mathscr{C}, \leqslant)$ such that $(E, \mathscr{C})$ is a matroid and $\leqslant$ is a linear order on E. For two ordered matroids $\mathscr{M}_1 = (E, \mathscr{C}_1, \leqslant_1)$ and $\mathscr{M}_2 = (E, \mathscr{C}_2, \leqslant_2)$ on the same ground set, a subset K of E is an *$\mathscr{M}_1\mathscr{M}_2$-kernel*, if K is independent in both matroids $(E, \mathscr{C}_1)$ and $(E, \mathscr{C}_2)$, and for any element e in $E \setminus K$ there is a subset C_e of K and an index $i = 1, 2$ so that

$$\{e\} \cup C_e \in \mathscr{C}_i \quad \text{and} \quad e \leqslant_i c \text{ for any } c \in C_e.$$

Theorem 2.5 (see Fleiner [3,4]). *For any pair $\mathscr{M}_1, \mathscr{M}_2$ of ordered matroids there exists an $\mathscr{M}_1\mathscr{M}_2$-kernel.*

Let $\mathscr{M}_1, \mathscr{M}_2, \dots, \mathscr{M}_k$ be ordered matroids on the same ground set E, where $\mathscr{M}_i = (E, \mathscr{C}_i, \leqslant_i)$. A vector $x \in \mathbb{R}^E_+$ is called a *fractional kernel for $\mathscr{M}_1, \mathscr{M}_2, \dots, \mathscr{M}_k$* if it satisfies the following two properties:

(1) x is *fractionally independent*, that is, $\sum_{e \in E'} x(e) \leqslant r_i(E')$ for any subset E' of E, where r_i is the rank function of the matroid $\mathscr{M}_i$.

(2) every element e of E is *fractionally optimally spanned* in one of the matroids, i.e., there exists a subset E' of E and a matroid $\mathscr{M}_i$, such that $e \leqslant_i e'$ for any $e' \in E'$, and $\sum_{e \in E'} x(e) = r_i(E' \cup \{e\})$.

Note that a fractional matroid-kernel for two matroids that happens to be integral is a matroid kernel.

Theorem 2.6. *Every family $\mathscr{M}_i = (E, \mathscr{C}_i, \leqslant_i)(i = 1, 2, \dots, k)$ of ordered matroids has a fractional kernel.*

Proof. Let B' be a matrix whose rows are indexed by pairs (i, F), where $1 \leqslant i \leqslant k$ and $F \subseteq E$, and whose columns are indexed by E, the $((i, F), e)$ entry being 1 if $e \in F$, 0 otherwise. Let $B := [I, B']$.

Define matrix C' on the same row and column sets as those of B', by letting its $((i, F), e)$ entry be the height of e in $\leqslant_i$ if $e \in F$ and $|F| + 1$ otherwise. Append an appropriate matrix on the left of C', so as to get a matrix C as in Theorem 1.1. Let b be the vector on E defined by $b_{(i,F)} := r_i(F)$.

Apply Theorem 1.1 to B, C and b. Let x be the vector whose existence is guaranteed in the theorem and x' be the restriction of x to E. We claim that x' is a fractional kernel for our matroids. As $Bx = b$ and both B and x are nonnegative, we have $B'x' \leqslant b$. In other words, x' is fractionally independent. The domination property of $\mathrm{supp}(x)$ yields that for any element e of E there is a subset F and a matroid $\mathscr{M}_i$ such that we have

$$e \leqslant_i f \quad \text{for any element } f \text{ of } F' := F \cap \mathrm{supp}(x). \tag{1}$$

Since $c_{(i,F),(i,F)}$ is smallest in row (i, F) of C, column (i, F) does not belong to $\mathrm{supp}(x)$. Thus $(Bx)_{(i,F)} = r_i(F)$ implies

$$r_i(F') \geqslant \sum_{f \in F'} x'(f) = \sum_{f \in F} x(f) = r_i(F) \geqslant r_i(F').$$

In particular, $F' \neq \emptyset$, hence (1) and the definition of $\leqslant_i$ shows that $e \in F$, and this proves the optimal spanning property of x'. $\square$

Remark. Theorem 2.6 is indeed a generalisation of Theorem 2.5, by Edmonds' characterisation of the matroid intersection polytope. Theorem 2.3 does not follow in this way from Theorem 2.4. See [1] for the details.

3. A link with a theorem of Shapley

A *simplicial complex* is a non-empty family $\mathscr{C}$ of subsets of a finite ground set such that $A \subset B \in \mathscr{C}$ implies $A \in \mathscr{C}$. Members of $\mathscr{C}$ are called *simplices* or *faces*. Let us call a simplicial complex *manifold-like* if, denoting its rank by n (that is, the maximal cardinality of a simplex in it is $n + 1$), every face of cardinality n in it is contained in two faces of cardinality $n+1$. The *dual* $\mathscr{D}^*$ of a complex $\mathscr{D}$ is the set of complements of its simplices. Just like in the case of complexes, members of a dual complex are also called *faces*.

Lemma 3.1. *If $\mathscr{C}, \mathscr{D}$ are two manifold-like complexes on the same ground set X, then the number of maximum cardinality faces of $\mathscr{C}$ that are also minimum cardinality faces of $\mathscr{D}^*$ is even.*

Proof. Let $\mathscr{C}_{\max}$ be the family of faces of $\mathscr{C}$ of maximum cardinality and $\mathscr{D}^*_{\min}$ be the set of faces of $\mathscr{D}^*$ of minimum cardinality. We may clearly assume that these two cardinalities are equal, as otherwise the lemma claims the triviality that zero is an even number. Fix an element x of X, and define an auxiliary digraph $\vec{G}$ on $\mathscr{C}_{\max} \cup \mathscr{D}^*_{\min}$ by drawing an arc from $C \in \mathscr{C}_{\max}$ to $D \in \mathscr{D}^*_{\min}$ if $D \setminus C = \{x\}$.

Let $D \in \mathscr{D}^*_{\min}$. If $x \notin D$ or $D \setminus \{x\} \notin \mathscr{C}$ then no arc enters D in $\vec{G}$. Otherwise, as $\mathscr{C}$ is manifold-like, there are exactly two different members C_1, C_2 of $\mathscr{C}_{\max}$ of the form $C_i = D \setminus \{x\} \cup \{y_i\}$ for some elements y_1, y_2 of X. If $y_1 \neq x \neq y_2$ then the in-degree of D in $\vec{G}$ is two and D is not a member of $\mathscr{C}_{\max}$. Else D has in-degree exactly one, and $x \in D \in \mathscr{C}_{\max} \cap \mathscr{D}^*_{\min}$.

Similarly, let $C \in \mathscr{C}_{\max}$. If $x \in C$ or $C \cup \{x\} \notin \mathscr{D}^*$ then no arc of $\vec{G}$ leaves C. Otherwise, $\mathscr{D}$ being manifold-like, there are exactly two members D_1, D_2 of $\mathscr{D}^*_{\min}$ of the form $D_i = C \cup \{x\} \setminus \{y_i\}$ for some elements y_1, y_2 of X. If $y_1 \neq x \neq y_2$ then the out-degree of C is exactly two and C is not a member of $\mathscr{D}^*_{\min}$. Else the out-degree of C in $\vec{G}$ is exactly one and $x \notin C \in \mathscr{C}_{\max} \cap \mathscr{D}^*_{\min}$.

Let G be the underlying undirected graph of $\vec{G}$. The above argument shows that a vertex v of G has degree zero or two if $v \in \mathscr{C}_{\max} \Delta \mathscr{D}^*_{\min}$ and v has degree one if $v \in \mathscr{C}_{\max} \cap \mathscr{D}^*_{\min}$. As the number of odd degree vertices of a finite graph is even, the lemma follows. $\square$

What examples are there of manifold-like complexes? Of course, a triangulation of a closed manifold is of this sort. (We call this complex a *manifold-complex*.) Another well-known example of a dual manifold-like complex is the *cone complex*: let X be a set of vectors in $\mathbb{R}^n$, and b be a vector not lying in the positive cone spanned by any $n-1$ elements of X. Consider the set $\mathscr{C}^* := \{A \subseteq X\colon b \in \text{cone}(A)\}$. It is a well-known fact from linear programming that if $b \in \text{cone}(A)$, where $A \subset X, |A| = n$ and $z \in X \setminus A$, then there exists a unique element $a \in A$ such that $b \in \text{cone}(A \cup \{z\} \setminus \{a\})$, that is, $\mathscr{C}^*$ is indeed a dual manifold-like complex.

A third example of a manifold-like complex is the *domination complex*. Let C be a matrix as in Theorem 1.1 with the additional property that in each row of C all entries are different. Then it is not difficult to check that the family of dominating column sets together with the extra member $[n]$ is a manifold-like complex. (For the details, see [2].)

Applying Lemma 3.1 to a manifold complex of the n-dimensional sphere and a cone complex yields a generalisation of Sperner's lemma.

Lemma 3.2. *Let the vertices of a triangulation of the sphere S^n be labelled with vectors from $\mathbb{R}^{n+1}$. Let $b \in \mathbb{R}^{n+1}$ be a vector that is not belonging to the cone spanned by fewer than $n+1$ labels. Then there are an even number of simplices S of the triangulation with the property that b is in the cone spanned by the vertex-labels of S.*

Sperner's lemma is obtained by taking the labels to be the standard unit vectors $(0, 0, \dots, 1, \dots, 0)$ and $b = \mathbf{1}$ is the all 1's vector. This is not the standard way the lemma is stated, but is well known to be equivalent to it, see e.g. [7]. Lemma 3.2 is undoubtedly known, but we do not know a reference to it. Shapley [9] proved it for the case that the labels are 0, 1 vectors, but his proof works also for general vectors.

By a general position argument, the proof of Scarf's lemma is a straightforward application of Theorem 3.1 to a domination complex and a cone complex.

Shapley's theorem is proved via Brouwer's fixed point theorem (which is also easily implied by it). This, and the similarity between its proof and the proof of Scarf's lemma, suggests that perhaps there is a fixed point theorem related to the latter. A supporting fact is that in [4] there was given a proof of Gale–Shapley's theorem using the Knaster–Tarski fixed point theorem for lattices.

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