LINEAR PROGRAMMING

1. Trading

A trader is trying to make as much money as possible by transporting goods. The budget is 10 000 000 Kč and the maximal weight of the load is 1000 kg. Design a linear program to maximize their profit.

Material	Price per kilo	Profit per kilo
Gold	100 000	100
Silver	14 000	200
Spices	4 000	150
Cocoa	700	10

2. More linear programs

Formulate linear programs for the following problems in a directed graph:

a) Shortest $s - t$ path

b) Maximal $s - t$ flow

c) Minimal $s - t$ cut

Formulate linear programs for the following problems in undirected graphs.

d) Maximum matching

e) Minimal vertex cover

Also determine:

I. What are the relationships between the individual linear programs?

II. What is the relationship between the optimum of the integer program and its relaxations for the programs a), b) a c)?

Hint: Their constraint matrix has a special form with unique properties.

3. Load balancing

Formulate an integer linear program for scheduling / load balancing on identical machines.

OTHER PROBLEMS

4. Parallel coloring

- A Las Vegas algorithm is a randomized algorithm that always answers correctly, but the running time is random for each run and its expected value is finite.

Find a Las Vegas parallel algorithm that uses n processors to properly color a graph with n vertices and maximum degree Δ , using 2Δ colors in expected time $O(\Delta \log n)$.

5. k-Center problem

- We define $R(S) := \max_{v \in V} d(v, S)$, where $d(v, S) := \min_{s \in S} d(v, s)$ for any $S \subset V,\ v \in V$.
- In the k-Center problem:
  - **Input** : We are given a metric space (V, d) and $k \in \mathbb{N}$.
  - **Output** : We want to find $S \subset V$ minimizing $R(S)$ such that $|S| \leq k$.

Find a 2-approximation algorithm the k-Center problem.

6. Graph balancing

We are given an unoriented graph with non-negative weights on the edges. Given an orientation, we define the weight of a vertex as the sum of weights of the incoming edges. The goal is to orient all the edges to minimize the weight of the heaviest vertex.

- Formulate an ILP and round the relaxation to get a 2-approximation algorithm.

7. Maximal directed cut

We are given an oriented graph with non-negative weights on the edges. The goal is to find $S \subseteq V$ such that the weight of edges from S to $V \backslash S$ is maximized.

- Formulate a (trivial) $\frac{1}{4}$-approximation algorithm.
- Formulate an ILP for this problem, relax it and add a vertex v to S with probability $\frac{1}{4} + \frac{x_v}{2}$. Did you get a $\frac{1}{2}$-approximation?

8. Integrality gap

Let OPT and OPT_{LP} be the optimal values (of an instance) of an integer LP and its relaxation respectively. The **integrality gap** of this linear program (instance) is $\frac{\text{OPT}}{OPT_{LP}}$.

- Show that MAX-SAT has integrality gap at least $\frac{4}{3}$.
- Show that vertex cover has integrality gap at least $2(1 - \frac{1}{n})$.