1. Repeated coin flips

We flip a fair coin (probability of heads or tails is 0.5) eight times. Determine the probability of the following events:

a) The number of heads is equal to the number of tails.
b) The number of heads is greater than the number of tails.
c) For $i \in [4]$, the i -th and $(9-i)$ -th flips have the same result.
d) There are at least four heads in a row.

2. Simulating a fair coin with a biased one

We flip a biased coin which lands on heads with probability p .

a) How do we simulate a single flip of a fair coin by flipping the biased coin multiple times?
b) What is the expected number of flips needed?

3. Simulating a biased coin with a fair one

We flip a fair coin repeatedly to simulate one landing on heads with probability $p \in \mathbb{R}$. How do we proceed? Determine the number of flips needed.

4. Useful bounds

a) Prove $n! \leq 2^{n \log_2 n}$.
b) Prove $(1-p)^n \leq e^{-pn}$ for $p \in \mathbb{R}$.

5. Complexity classes

Definitions of the probability complexity classes are on the other side.

a) Show that the following definition is equivalent to the definition of ZPP:
   - Always runs in polynomial time.
   - **TRUE** instance is accepted by the algorithm with probability at least $\frac{1}{2}$, otherwise it says "don't know".
   - **FALSE** instance is rejected by the algorithm with probability at least $\frac{1}{2}$, otherwise it says "don't know".
b) Prove the following inclusions:
   - P $\subseteq$ ZPP
   - RP $\subseteq$ NP
   - ZPP $=$ RP $\cap$ co-RP
   - RP $\cup$ co-RP $\subseteq$ BPP

Definitions

- A problem is in **BPP** if $\exists$ algorithm satisfying:
  - Always runs in polynomial time.
  - If the correct answer is **TRUE** , algorithms accepts with probability at least $\frac{2}{3}$.
  - If the correct answer is **FALSE** , algorithms accepts with probability at most $\frac{1}{3}$.
- A problem is in **RP** if $\exists$ algorithm satisfying:
  - Always runs in polynomial time.
  - If the correct answer is **TRUE** , algorithms accepts with probability at least $\frac{1}{2}$.
  - If the correct answer is **FALSE** , algorithms always rejects.
- A problem is in **ZPP** if $\exists$ algorithm satisfying:
  - Only the expectation of the running time is polynomial.
  - Always answers correctly.