On Polynomial-Time Combinatorial Algorithms for Maximum L-Bounded Flow

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Length Bounded Flow Problem

L-bounded flow

- a flow decomposable into flow paths of length at most L

Input

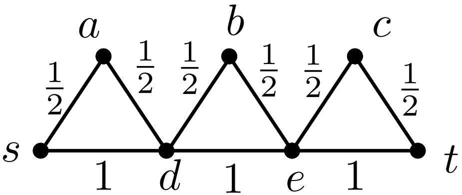
- graph $G = (V, E)$
- source-sink pair $s, t \in V$
- integer parameter L

Output and Objective

- find an L-bounded flow of maximum size

Example of a Length Bounded Flow

$L = 4$



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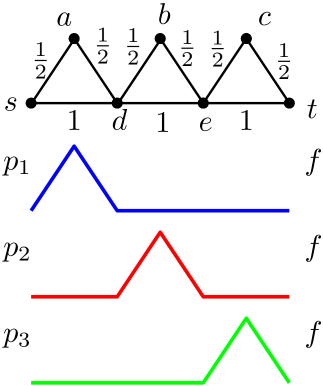

$f(p_1) = \frac{1}{2}$

$f(p_2) = \frac{1}{2}$

$f(p_3) = \frac{1}{2}$

Example of a Maximum Length Bounded Flow

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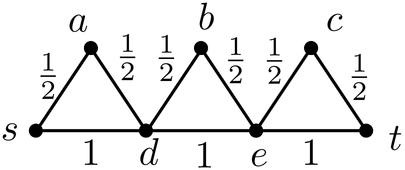


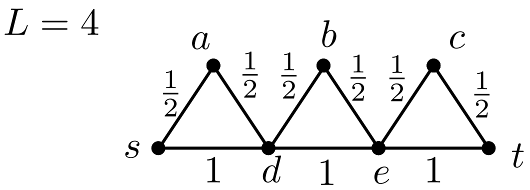
Observe

- every L-bounded path has to use at least two bottom edges
- three bottom edges $\Rightarrow$ max L-bounded flow at most $\frac{3}{2}$

Takeaway

- The maximum L-bounded flow need not be integral, even on graphs with unit capacities.

Example of a Maximum Length Bounded Flow



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Non-Example of a Length Bounded Flow

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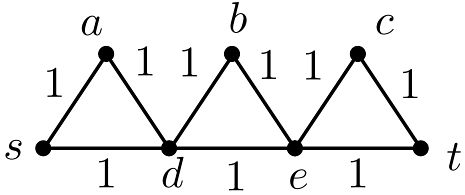

4-bounded flow?

- No - it's bigger than the maximum 4-bounded flow.

5-bounded flow?

- Yes - decompose into two paths of length 4 and 5.

Non-Example of a Length Bounded Flow

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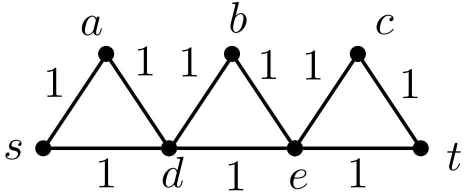

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Length Bounded Cut Problem

Input

- graph $G = (V, E)$
- source-sink pair $s, t \in V$
- integer parameter L

Output and Objective

- a subset of edges $F \subseteq E$ such that in $G \setminus F$, the distance between s and t is at least $L + 1$
- find an L-bounded cut of minimum size

Also known as

- Short paths interdiction problem, and
- Most vital edges for shortest paths problem

Related Results

Length Bounded Flows and Cuts

- 1971 - Adámek and Koubek - introduction of L-bounded flows and cuts; duality does not hold
- 1981 - Koubek and Říha - combinatorial algorithm for the maximum length bounded flow *flawed*
- 1995 - Bar-Noy et al. - L-bounded cut NP-hard
- 2002 - K. and Scheideler - L-bounded flow in P, by poly-size LP
- 2003 - Baier - FPTAS for L-bounded flow with edge lengths, using the ellipsoid algorithm
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Related Problems

Shortest Hop Constrained Path

- find the shortest path between two vertices, wrt edge lengths, with a bounded number of edges (hops)

L-Bounded Disjoint Paths

- find the maximum number of disjoint paths between two vertices, each of a bounded length

Most Vital Edges for Shortest Paths Problem

- given an integer k, find a subset of k edges whose removal maximizes the distance between s and t

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Our Contribution

Length Bounded Flow

- The algorithm of Koubek and Říha is not correct
- A combinatorial FPTAS for the maximum L-bounded flow, i.e., $(1+\varepsilon)$ approximation of OPT in time $(\varepsilon^{-2}|E|^2 L \log L)$
- A combinatorial FPTAS for the NP-hard maximum L-bounded flow with edge lengths

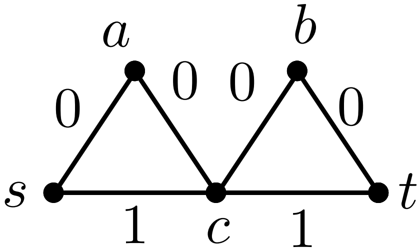
Open Problem

- Design a poly-time combinatorial algorithm for the maximum L-bounded flow

Combinatorial = the algorithm does not explicitely use LP and linear algebra methods

The Algorithm of Koubek and Říha

$L = 3$



Example

- Not a maximum L-bounded flow.
- No space for adding a new L-bounded $s - t$ path.
- By diverting the flow on $c - t$ along $c - b - t$, we obtain space for a new L-bounded $s - t$ path $s - a - c - t$.

The Algorithm of Koubek and Říha, cont'd.

Main Idea

- Given an L-bounded flow f and its decomposition, describe by a tree structure how to combine segments of paths from the flow f with segments of empty edges into a larger L-bounded flow.

Technical Details

- Many ...
- Define a tree called *increasing L-system* - generalization of an augmenting flow.
- Various types of nodes: for diverting flow, for shortening flow, pointers to other nodes, etc.
- Each node has a plenty of attributes to take care about the length bounds and flow conservation at each vertex.

The Algorithm of Koubek and Říha, cont'd.

Proof Structure

1. If f is not maximal L-bounded flow, then there exists an increasing L-system.
2. If there exists an increasing L-system, then it is possible to obtain a larger L-bounded flow.

$\Rightarrow$ iterative improvements possible

Cf. Ford-Fulkerson alg. for classical flow: if there is an augmenting path in the residual network, increase the flow along it

Difficulty

- The second claim does not hold:
  the existence of an increasing L-system does not imply the possibility to increase the flow!

The Algorithm of Koubek and Říha, cont'd.

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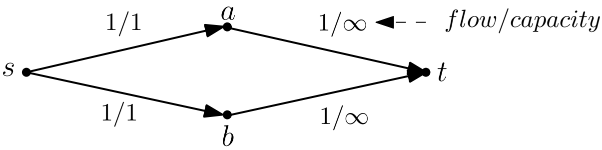
Example:

For the following graph and the maximum L-bounded flow, ...



The Algorithm of Koubek and Říha, cont'd.

... an increasing *L*-system exists.

$u_0 :\ 1 - son$	$\overline{q}(u_0) = \{s, a, t\}$
$q(u_0) = \emptyset$	$saturated\ edge = \{sa\}$
$u_1 :\ 3 - son$	$saturated\ edge = \{sb\}$
$q(u_1) = \emptyset$	$h(u_1) = \{sa\}$
$\overline{q}(u_1) = \{s, b, t\}$	$j(u_1) = 1$
$u_2 :\ 3 - son$	$saturated\ edge = \{sa\}$
$q(u_2) = \emptyset$	$h(u_2) = \{sb\}$
$\overline{q}(u_0) = \{s, a, t\}$	$j(u_2) = 2$
$u_3 :\ 4 - son$	$h(u_3) = \{sa\}$
	$o(u_3) = u_1$

Informally, the pointer nodes in the tree may create a deadlock cycle: every node is expecting from some other node to do the job (of diverting some flow) but nobody does it.

Part II

Notation

- $\mathcal{P}_L$ - the set of simple paths between s and t of length at most L
- for $P \in \mathcal{P}_L$, $x(p)$ - a variable that expresses the flow on P
- for $e \in E$, $c(e)$ - the capacity of the edge e

Consider the path based LP formulation of the maximum L-bounded flow, and its dual:

$$\begin{aligned} \max \quad & \sum_{P \in \mathcal{P}_L} x(P) \\ \text{s.t.} \quad & \sum_{\substack{P \in \mathcal{P}_L: \\ e \in P}} x(P) \le c(e) \quad \forall e \in E \\ & x \ge 0 \end{aligned}$$

$$\begin{aligned} \min \quad & \sum_{e \in E} c(e) y(e) \\ \text{s.t.} \quad & \sum_{e \in P} y(e) \ge 1 \quad \forall P \in \mathcal{P}_L \\ & y \ge 0 \end{aligned}$$

FPTAS for Maximum L-bounded Flow, cont'd.

Algorithm

- iteratively construct (to-be) solutions for both primal and dual:
- an L-bounded flow x (may violate the capacities, initially $x = 0$),
- a length y on the edges (initially $y(e) = \delta(\varepsilon)$ for each e)

In each iteration

- find a y-shortest L-bounded path $P \in \mathcal{P}_L$
- route c units of flow on P, where $c = \min_{e \in P} c(e)$
- for $e \in P$, update the lengths: $y(e) := y(e)(1 + \varepsilon \frac{c}{c(e)})$

Termination

- stop when the y-shortest path P is longer than 1
- down-scale x to satisfy all capacity constraints

FPTAS for Maximum L-bounded Flow, cont'd.

APPROX(ε)

1: $y(e) \leftarrow \delta(\varepsilon) \quad \forall e \in E,\ x(P) \leftarrow 0 \quad \forall P \in \mathcal{P}_L$
2: **while** the y-shortest L-bounded s-t path has length < 1 **do**
3: $\quad P \leftarrow$ the y-shortest L-bounded s-t path
4: $\quad c \leftarrow \min_{e \in P} c(e)$
5: $\quad x(P) \leftarrow x(P) + c$
6: $\quad y(e) \leftarrow y(e)(1 + \varepsilon c / c(e)) \quad \forall e \in P$
7: **end while**
8: **return** x

Intuition

- make edges with large flow long
- send flow along short paths, i.e., avoid heavily loaded edges

Note: The y-shortest L-bounded s-t path can be computed by a modification of Dijkstra's algorithm.

FPTAS for Maximum L-bounded Flow, cont'd.

Remarks

- The same structure as in the algorithm for maximum multicommodity flow by Garg and Könemann (2007).
- Example of the Multiplicative Weights Update Method.

FPTAS for Maximum L-bounded Flow, cont'd.

Lemma

The flow x scaled down by a factor of $\log_{1+\varepsilon}\frac{1+\varepsilon}{\delta}$ *is feasible.*

Proof.

Consider an edge $e \in E$ and let $f(e)$ be the final flow on e.
Iterations $i_1, \ldots, i_r$ contributed to $f(e)$ by $c_1, \ldots, c_r$, i.e., $f(e) = \sum_{j=1}^{r} c_j$.
At the end: $1 + \varepsilon > y(e)$.
Thus,

$$1 + \varepsilon > y(e) = \delta \prod_{j=1}^{r}(1 + \varepsilon \frac{c_j}{c(e)}) \geq \delta \prod_{j=1}^{r}(1 + \varepsilon)^{\frac{c_j}{c(e)}} = \delta(1 + \varepsilon)^{\frac{f(e)}{c(e)}} ,$$

which implies

$$\log_{1+\varepsilon}\frac{1+\varepsilon}{\delta} \geq \frac{f(e)}{c(e)} .$$

□

FPTAS for Maximum *L*-bounded Flow, cont'd.

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Theorem

For every $0 < \varepsilon < 1$, *the algorithm computes an* $(1 + \varepsilon)$*-approximation to the maximum L-bounded flow in time* $(\varepsilon^{-2} m^2 L \log L)$.

Extensions

Generalized Setting

- With some adjustments, the FPTAS works even in the NP-hard setting with edge lengths.

Differences

- difficulty: finding the y-shortest L-bounded path (a procedure of the FPTAS) is NP-hard if edges have lengths
- use an approximately y-shortest L-bounded path instead

L-bounded Flow - State of Art

- LP algorithm - *OPT* in poly time
- Combinatorial algorithm - $(1+\varepsilon)$ approx. in time $(\varepsilon^{-2}|E|^2 L \log L)$

Open Problem

- Design an exact poly-time combinatorial algorithm for the maximum L-bounded flow.

Thank you!