Extension Complexity, MSO Logic, and Treewidth

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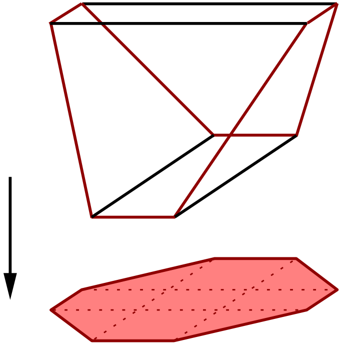
Extended Formulation of a Polytope P

Definitions

- A polytope $Q \subseteq \mathbb{R}^{d+r}$ is an extended formulation of $P \subseteq \mathbb{R}^d$ if P is a projection of Q onto the first d coordinates.
- The size of P is the number of its facet-defining inequalities.
- The extension complexity of a polytope P, denoted by $\mathrm{xc}(P)$, is the size of its smallest extended formulation.



Extended Formulations

What is the meaning?

- Complexity measure

Selected related results

- 1986-87 - Swart, attempts to prove P=NP by giving polynomial size LP for TSP
- 1988 - Yannakakis, every symmetric EF for TSP has exponential size
- 2007 - Sellmann et al., EF for CSP of size $O(n^{\tau})$ for graphs of treewidth τ using Sherali-Adams hierarchy
- 2012 - Fiorini et al., no polynomial-size EF for TSP
- 2014 - Rothvoß, no polynomial-size EF for matching polytope
- 2015 - K., Koutecký - EF for CSP of size $O(D^{\tau}n)$ for graphs of treewidth τ - includes vertex cover, independent set ...
- ...

MSOL Polytope

Input

- a graph $G = (V, E)$ with n vertices and treewidth τ
- an MSOL formula $\varphi(\vec{X})$ with m free set variables $X_1, \ldots, X_m$

MSOL polytope

$$P_\varphi(G) = \operatorname{conv}\left(\{y \in \{0,1\}^{nm} \mid y \text{ satisfies } \varphi\}\right) .$$

where $y_v^i = 1$ represents $v \in X_i$ and $y_v^i = 0$ represents $v \notin X_i$.

Question

- What is the extension complexity of $P_\varphi(G)$?

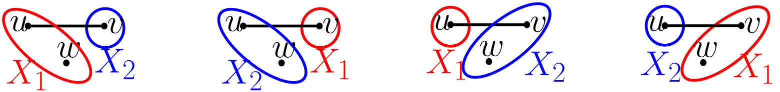
Example

Formula and Graph

- $\mathrm{2COL}(X_1, X_2) = (X_1 \cap X_2 = \emptyset) \wedge \forall x (x \in X_1 \vee x \in X_2) \wedge$
  $(\forall x \in X_1 \forall y \in X_1, x \neq y \rightarrow \neg E(x, y)) \wedge$
  $(\forall x \in X_2 \forall y \in X_2, x \neq y \rightarrow \neg E(x, y))$
- $G = (\{u, v, w\}, \{\{u, v\}\})$

Variables

- $(y_u^1, y_u^2, y_v^1, y_v^2, y_w^1, y_w^2)$



$(1, 0, 0, 1, 1, 0)$ $(0, 1, 1, 0, 0, 1)$ $(1, 0, 0, 1, 0, 1)$ $(0, 1, 1, 0, 1, 0)$

$P_\varphi(G) =$
$\mathrm{conv}(\{(1, 0, 0, 1, 1, 0), (0, 1, 1, 0, 0, 1), (1, 0, 0, 1, 0, 1), (0, 1, 1, 0, 1, 0)\})$

Main Result

Theorem (K., Koutecký, Tiwary, 2016)

For every graph G on n vertices with $tw(G) = \tau$ and for every MSOL formula φ,

$$\mathrm{xc}(P_\varphi(G)) = f(|\varphi|, \tau) \cdot n$$

where f is some computable function.

As a corollary, it yields the famous result about LinEMSOL problems:

Theorem (Arnborg, Lagergren, and Seese, 1991)

Every LinEMSOL problem is solvable in polynomial time for graphs of bounded treewidth.

Remarks

Theorem (Courcelle, 1990)

Every graph property definable in MSOL is decidable in linear time for graphs of bounded treewidth.

Our theorem: Not a surprising result — on the high level

Merging common wisdom from various CS areas

- Courcelle's theorem = dynamic programming (parameterized complexity)
- dynamic programming = compact extended formulation (polyhedral combinatorics)

Our theorem: Far from obvious — when it comes to details

- no black-box results for the above knowledge

Our Tool: Gluing Polytopes

The *cartesian product* of two polytopes P_1 and P_2

$$P_1 \times P_2 = \operatorname{conv}\{(x, y) \mid x \in \operatorname{vert}(P_1), y \in \operatorname{vert}(P_2)\}$$

The *glued product* of $P \in \mathbb{R}^{d_1+k}$ and $Q \in \mathbb{R}^{d_2+k}$, with respect to the last k coordinates

$$P \times_k Q = \operatorname{conv}\left\{(x, y, z) \in \mathbb{R}^{d_1+d_2+k} \mid (x, z) \in \operatorname{vert}(P), (y, z) \in \operatorname{vert}(Q)\right\}$$

Lemma (Gluing lemma, Margot 1994, KKT 2016)

Let P and Q be $0/1$-polytopes and let the k glued coordinates in P be labeled z, and the k glued coordinates in Q be labeled w.
If $\mathbf{1}^\intercal z \leqslant 1$ is valid for P and $\mathbf{1}^\intercal w \leqslant 1$ is valid for Q, then $\operatorname{xc}(P \times_k Q) \leqslant \operatorname{xc}(P) + \operatorname{xc}(Q)$.

Tree decomposition of $G = (V, E)$

a tree T, each node $a \in T$ has an assigned set of vertices $B(a) \subseteq V$, called a bag, some properties ...

Nice tree decomposition

- Leaf node: a leaf a of T with $B(a) = \emptyset$.
- Introduce node: an internal node a of T with one child b for which $B(a) = B(b) \cup \{v\}$ for some $v \in B(a)$.
- Forget node: an internal node a of T with one child b for which $B(a) = B(b) \setminus \{v\}$ for some $v \in B(b)$.
- Join node: an internal node a with two children b and c with $B(a) = B(b) = B(c)$.

Subgraph of G induced by a tree node

For a node $a \in V(T)$, we denote by T_a the subtree of T rooted in a, and by G_a the subgraph of G induced by all vertices in bags of T_a.

Rough Sketch of the Proof

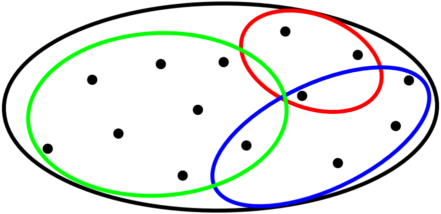
Idea

For a formula φ, a graph G and a nice tree decomposition T of G

- for every node a of T define a small polytope representing assignments of the bag vertices to the sets (i.e., free variables) that are extendible to a feasible assignment of all vertices in G_a
- process the tree T in a bottom up fashion and glue the small polytopes by Gluing lemma
- as every polytope is small, by Gluing lemma the polytope in the root of T is of size $O(n)$
- show that $P_\varphi(G)$ is a projection of the polytope in the root of T

$[m]$-colored graph

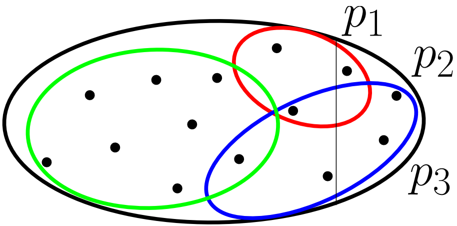
a pair $(G, \vec{V})$ where $G = (V, E)$, $\vec{V} = (V_1, \ldots, V_m)$ and $V_i \subseteq V$



Colored and Boundaried Graphs

$[m]$-colored τ-boundaried graph

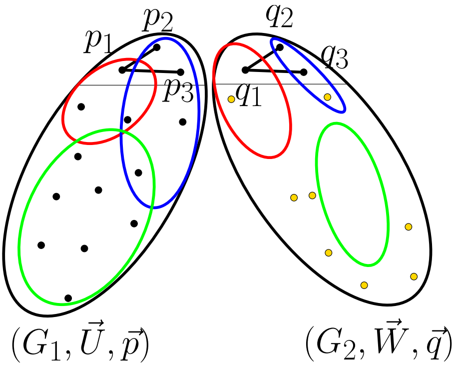
a triple $(G, \vec{V}, \vec{p})$ where $(G, \vec{V})$ is an $[m]$-colored graph and $\vec{p} = (p_1, \ldots, p_\tau)$ is a τ-tuple of vertices of G



Compatible Graphs

$(G_1, \vec{U}, \vec{p})$ and $(G_2, \vec{W}, \vec{q})$ are compatible

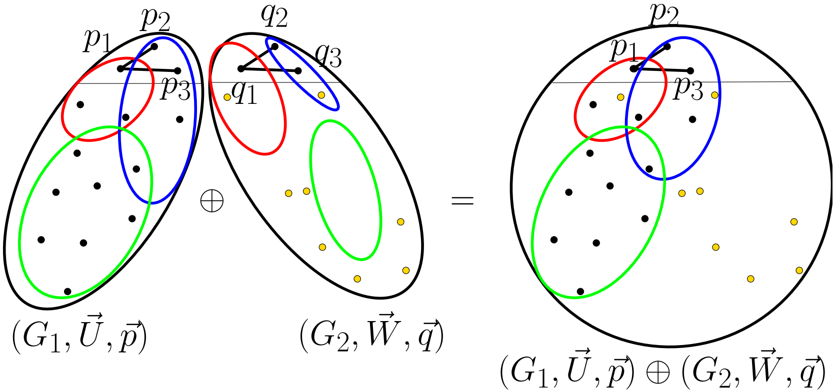
if subgraphs $G_1[\vec{p}]$ and $G_2[\vec{q}]$ are identical and colored the same way.



Join of Graphs

Join of compatible graphs $(G_1, \vec{U}, \vec{p})$ and $(G_2, \vec{W}, \vec{q})$

is $[m]$-colored τ-boundaried graph ...



Equivalence and Types of Graphs

$\mathrm{MSO}[k, \tau, m]$

all MSOL formulae over $[m]$-colored τ-boundaried graphs with $qr \leq k$

Equivalence $\equiv_k^{MSO}$

Two $[m]$-colored τ-boundaried graphs $G_1^{[m],\tau}$ and $G_2^{[m],\tau}$ are $\mathrm{MSO}[k]$-*equivalent* if they satisfy the same $\mathrm{MSO}[k, \tau, m]$ formulae.

Theorem (Libkin, 2004, implicitly in Courcelle, 1990)

For any fixed $\tau, k, m \in \mathbb{N}$, the equivalence relation $\equiv_k^{MSO}$ has a finite number of equivalence classes.

We denote the equivalence classes by $\mathcal{C} = \{\alpha_1 \ldots, \alpha_w\}$, fixing an ordering such that α_1 is the class containing the empty graph.

Type of a graph

Given an $[m]$-colored τ-boundaried graph, its type (w.r.t. $\equiv_k^{MSO}$) is the class to which it belongs.

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Join and Types

Lemma (Libkin, 2004)

If $G_a^{[m],\tau} \equiv_k^{MSO} G_{a'}^{[m],\tau}$ *and* $G_b^{[m],\tau} \equiv_k^{MSO} G_{b'}^{[m],\tau}$*, then*

$$\left(G_a^{[m],\tau} \oplus G_b^{[m],\tau}\right) \equiv_k^{MSO} \left(G_{a'}^{[m],\tau} \oplus G_{b'}^{[m],\tau}\right) \quad .$$

Meaning

The type of a join of two $[m]$-colored τ-boundaried graphs is determined by only a small amount of information about the two graphs, namely their types.

Feasible Types of Tree Decomposition Nodes

Feasible type of a node $b \in V(T)$

- every $\alpha \in \mathcal{C}$ such that there exist $X_1, \ldots, X_m \subseteq V(G_b)$: $(G_b, \vec{X}, B(b))$ is of type α where $\vec{X} = (X_1, \ldots, X_m)$
- Notation: $\mathcal{F}(b)$ - the set of feasible types of the node b where every type is represented by a binary vector $t_b \in \{0,1\}^{|\mathcal{C}|}$

Feasible triple of types for a join node c with children a, b

every triple $(\gamma_1, \gamma_2, \alpha)$ such that

- $\alpha \in \mathcal{F}(c)$, $\gamma_1 \in \mathcal{F}(a)$ and $\gamma_2 \in \mathcal{F}(b)$, and
- γ_1, γ_2 and α are mutually compatible,
- and there exist $\vec{X}^1, \vec{X}^2$ realizing γ_1 and γ_2 on a and b, such that $\vec{X} = (X_1^1 \cup X_1^2, \ldots, X_m^1 \cup X_m^2)$ realizes α on c.

Notation: $\mathcal{F}_t(c)$ - the set of feasible triples of types of the join node c.

Feasible pairs of types for a forget and introduce node c

analogously ... $\mathcal{F}_p(c)$

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The Construction

The basic polytopes

- b is a *leaf*: $P_b = \{\overbrace{100\ldots0}^{|\mathcal{C}|}\}$
- b is an *introduce* or *forget* node: $P_b = \operatorname{conv}(\mathcal{F}_p(b))$
- b is a *join* node: $P_b = \operatorname{conv}(\mathcal{F}_t(b))$

Lemma

Extension complexity of the polytopes P_b's is independent on n.

Proof: The sizes of the sets $\mathcal{F}(a)$, $\mathcal{F}_p(a)$, $\mathcal{F}_t(a)$ are independent on n.

Gluing them into larger polytopes

- b is a *leaf*: $Q_b = P_b$.
- b is an *introduce* or *forget* node. $Q_b = Q_a \times_{|\mathcal{C}|} P_b$
  where a is the child of b and the gluing is done along the coordinates t_a in Q_a and d_b in P_b.
- b is a *join* node. $Q_b = Q_a \times_{|\mathcal{C}|} P_b \times_{|\mathcal{C}|} Q_c$ where ....

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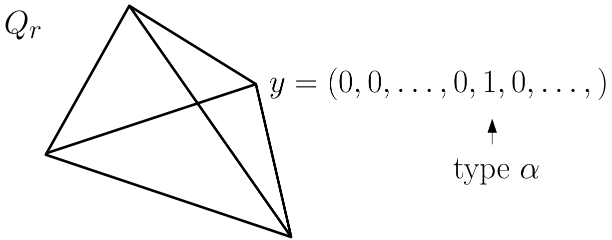
Lemma

For every node $b \in V(T)$ and every vertex y of the polytope Q_b there exist $X_1, \ldots, X_m \subseteq V(G_b)$ such that $(G_b, (X_1, \ldots, X_m), \sigma(B(b)))$ is of the type specified by the vector y.

Proof. By induction and previous Lemma.



Applying Lemma to the root of the decomposition tree and a few more steps completes the proof of the main theorem.

Final Remarks

Worth noting

- the extension complexity linear in the size of G
- optimization easy (LinEMSOL)
- the constructed polytope almost universal: apart from the last step (skipped), the construction depends on the *quantifier rank* of the formula only, not on the *formula* itself

Thank you!