Approximating Spanning Tree Congestion on Graphs with Polylog Degree

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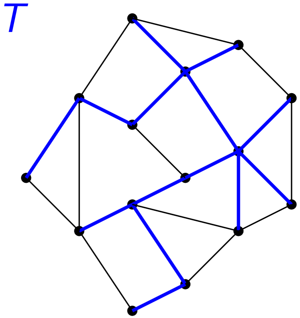
Preliminaries

Definitions

- *spanning tree* of $G = (V, E)$: *subgraph* of G that is a *tree* containing *all vertices* of G
- *cut* of $G = (V, E)$: partition of V into two subsets S and $V \setminus S$
- *cut size*: $|E(S, V \setminus S)|$ = number of edges between S and $V \setminus S$



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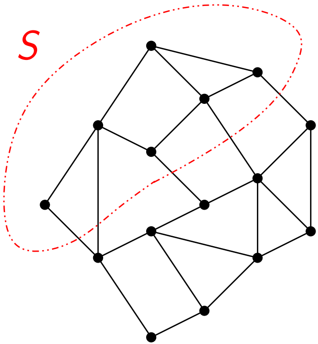

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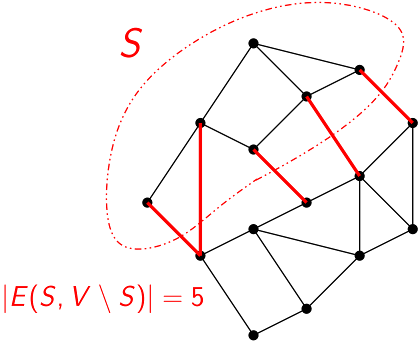


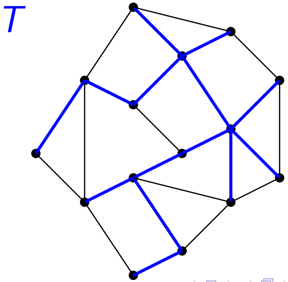
Spanning Tree Congestion

Cuts Induced by a Spanning Tree

- given a spanning tree T of G and an edge $e \in T$, the removal of e defines a cut in G - let $c_T(e)$ denote its size
- *congestion of span. tree T of G*: $C(G, T) = \max_{e \in T} c_T(e)$

Spanning Tree Congestion

- *spanning tree congestion of G*: $\mathrm{STC}(G) = \min_{T \in \mathcal{T}} C(G, T)$ where $\mathcal{T}$ is the set of all spanning trees of G



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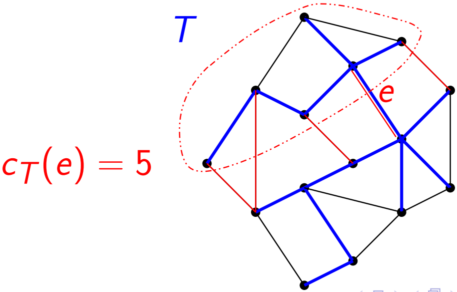

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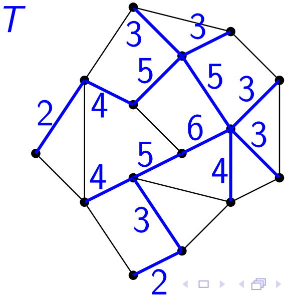

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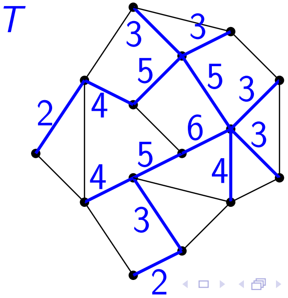
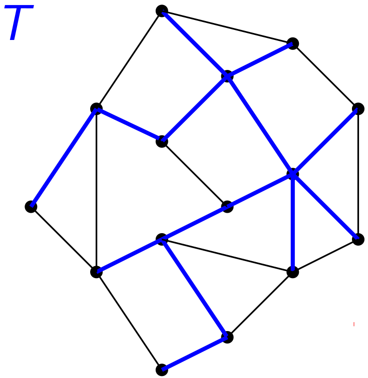
$STC(G) \leq 6$



Spanning Tree Congestion - Alternative View

Simulating G by its Spanning Tree T

- given a spanning tree T of G, for every edge $uv \in E(G)$ there is a unique path p_{uv} in T between u and v
- Observe: for every $e \in E(T)$, $c_T(e) = |\{uv \in E(G) \mid p_{uv} \ni e\}|$.



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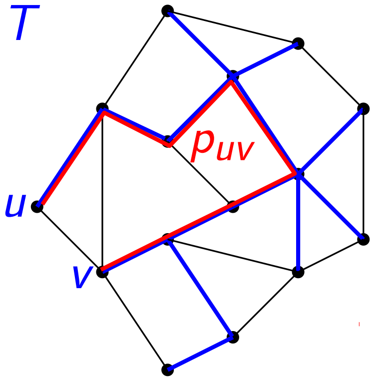

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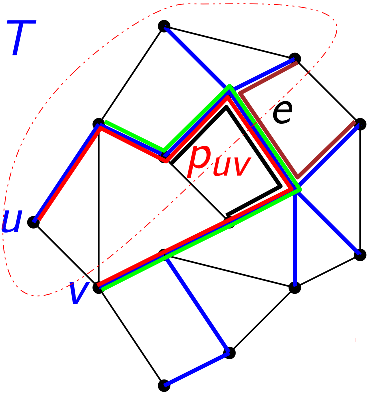

Spanning Tree Congestion - Problems

Spanning Tree Congestion

- given G, compute STC(G) and find the corresponding tree
- $k -$ STC: given G and $k \in \mathbb{N}$, is $\text{STC}(G) \leq k$?

Selected Known Results

- 1987 - Simonson: problem first studied, under different name
- 2004 - Ostrovskii: STC name, graph-theoretic results
- 2010 - Otachi et al., Löwenstein: NP-hard, even for planar graphs
- 2010 - Otachi et al.: for $k \leq 3$, $k-$ STC in P, $n/2$-approximation
- 2010 - Okamoto et al.: exact $O(2^n)$-time algorithm
- 2012 - Bodlaender et al.: NP-hard even for graphs with all but one degrees bounded by $O(1)$
  $8-$ STC NP-hard, no c-approximation, for $c < 9/8$
  $k-$ STC FPT w.r.t. k and max degree
- 2023 - Luu and Chrobak: $5-$ STC NP-hard, no c-approximation, for $c < 6/5$, unless P=NP

Note

- $\Omega(n)$ gap between lower and upper bounds on approximability

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New Results

Challenging Question

- Is there an $o(n)$-approximation?

Our Result

- $o(n)$-approximation for graphs with polylog degree

Tools

Bisection of $G = (V, E)$

- a partition of V into S and $V \setminus S$ such that $|S| = n/2$
- width of bisection: size of the corresponding cut - $|E(S, V \setminus S)|$
- minimum bisection of G: bisection of minimum width - $b(G)$

Theorem (Räcke, 2008)

There is an $O(\log n)$-approximation for minimum bisection.

Lemma (Folklore)

For every tree T of maximum degree Δ, there exists a bisection of width at most $\Delta \log n$.

New Lower Bounds

Key ingredients of our algorithm are two new STC lower bounds.

Lemma (Bisection Bound)

For every graph $G = (V, E)$ with n vertices and max degree Δ,

$$STC(G) \geq \frac{b(G)}{\Delta \log n} \,.$$

Lemma (Subgraph Bound)

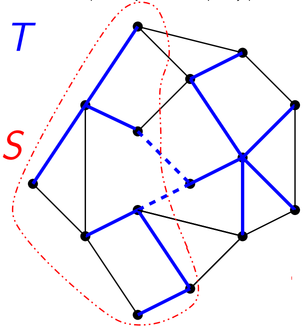
For every every graph $G = (V, E)$ and every connected $S \subseteq V$,

$$STC(G) \geq \frac{STC(G[S])}{|E(S, V \setminus S)|} \,.$$

Proof of Bisection Bound

Let T be the optimal spanning tree and $(S, V \setminus S)$ its minimum bisection. Then,

$$|E_T(S, V \setminus S)| \leq \Delta \log n \qquad \text{by Folklore Lemma,}$$
$$|E_G(S, V \setminus S)| \geq b(G) \qquad \text{by definition of bisection.}$$



For each $uv \in E_G(S, V \setminus S)$, the path p_{uv} uses at least one edge $e \in E_T(S, V \setminus S)$.

Thus, by Pigeonhole Principle, for at least one edge $e \in E_T(S, V \setminus S)$,

$$\frac{b(G)}{\Delta \log n} \leq c_T(e) \leq C(G, T) = \mathrm{STC}(G) . \quad \square$$

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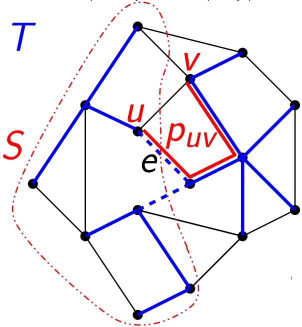

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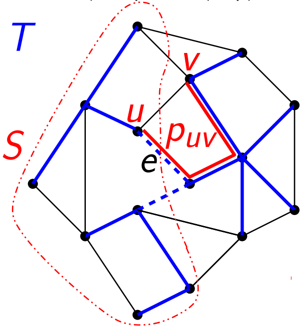

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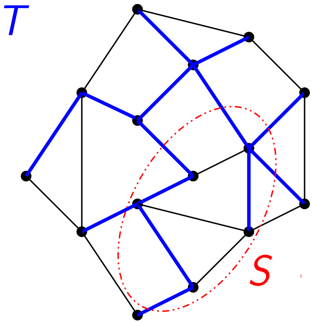

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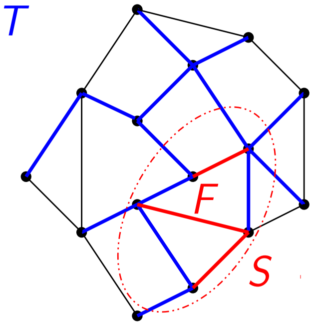

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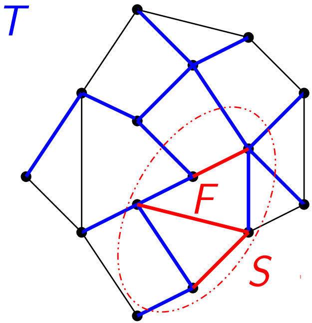

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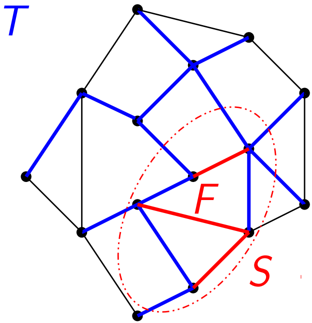

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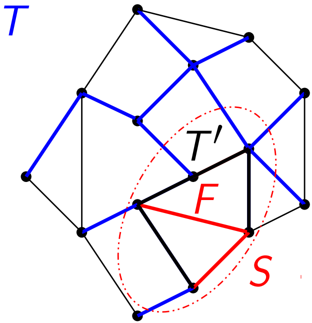

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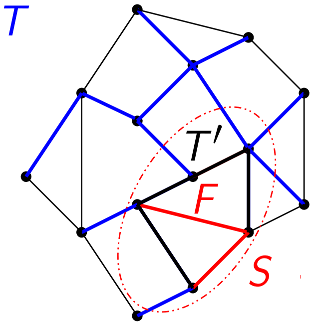

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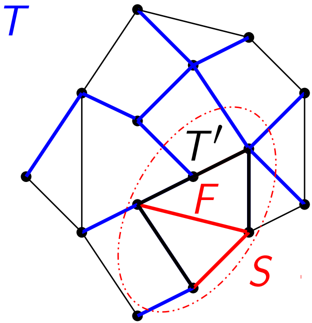

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Approximation Algorithm - Sketch

Ideas

- Recursively bisect the graph until each part is *small*.
- Either some of the bisections is *large*: then, by above lemmas, we have a strong lower bound, and any spanning tree is a good solution;
- Or all bisections are *small*: then the recursive decomposition yields a good solution, as the overall congestion is small (calculation needed);
- Note: The meaning of *large* depends on the recursion level - increases exponentially with the recursion level.

Approximation Algorithm

initial parameters $H = G$, $s = \frac{n}{2^{k-1}}$, $\sigma = 1$ where $k = \lceil \sqrt{\log n} \rceil + 1$

ConstructST(H, s, δ)

construct an approximate bisection $(S, V(H) \setminus S)$ of H
$F \leftarrow E(S, V(H) \setminus S)$; $b \leftarrow |F|$
if $b/\sigma \geq n^{1/k}$ or $|V(H)| \leq s$ **then**
  return any spanning tree of H
end if
for each connected component C of $H \setminus F$ **do**
  $T_C \leftarrow$ ConstructST$(C, s, \sigma + b)$
end for
arbitrarily connect all the spanning trees T_C by edges from F to form a spanning tree T of H
return T

Theorem

ConstructST *is an $\tilde{\mathcal{O}}(n^{1-1/k})$-approximation algorithm.*

Open Problems

Three STC Questions

- Is it possible to extend ConstructST for unbounded degrees?
- Is 4-STC solvable in polynomial time, or is it NP-complete?
- Is STC problem NP-hard for graphs with all degrees bounded by a constant? The known NP-hardness proofs require at least one vertex of unbounded degree.

Open Problems, cont'd.

Minimum Connected c-Cut

Given a graph $G = (V, E)$ and $c \in \mathbb{N}$, find a subset $S \subseteq V$ such that:

1. $|S| = c$,
2. the subgraph $G[S]$ is connected,
3. the cut size $|E(S, V \setminus S)|$ is the smallest among all subsets satisfying the other properties.

Let $f(G, c) = \min_{S \subset V}\{|E(S, V \setminus S)| : |S| = c \text{ and } G[S] \text{ connected}\}$.

Lemma (Law, Ostrowskii, 2010)

$$STC(G) \geq \min\left\{ f(G, c) \;\middle|\; \left\lceil \frac{n-1}{\Delta} \right\rceil \leq c \leq \frac{n}{2} \right\}.$$

If only any two of the three properties are required, then the problem is solvable, or at least reasonably approximable, in polynomial time.

Open Problem

- Design an approximation for the minimum connected c-cut.

Thank you!