SDP and c-balanced cuts - Illustration 19/12/2025

INPUT: $G=(V,E)$, where $V=\{1,\dots,n\}$, $0<c<\frac{1}{2}$

OUTPUT: c-balanced cut, i.e., $S\subseteq V$ s.t. $|S|\geq c\cdot n$, $|V\setminus S|\geq c\cdot n$

OBJECTIVE: $\min\ E(S,V\setminus S)=\delta(S)$

SDP relaxation: $\forall i\in V$, a vector variable $v_i\in\mathbb{R}^n$

$$\min \frac{1}{4}\sum_{\{i,j\}\in E}(v_i-v_j)^2$$

$$v_i^2=1 \qquad \forall i\in V$$

$$\frac{1}{4}\sum_{i<j}(v_i-v_j)^2\geq c\cdot(1-c)\cdot n^2$$

the Δ-inequality not needed here

$$\left((v_i-v_j)^2+(v_j-v_k)^2\geq(v_i-v_k)^2 \quad \forall i,j,k\in V\right)$$

Intuition: think about $v_i\in\{-1,1\}\ \forall i$

Key tool is the following procedure:

SET-FIND (c',σ,Δ)

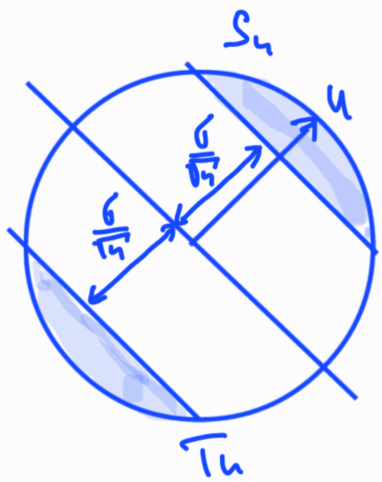
[We use: $c'=\frac{b}{8}$, $\sigma=\frac{b^2}{48}$, $\Delta=\frac{\sigma^2}{4\log n}$, where $b=0{,}878\cdot c\cdot(1-c)$]

INPUT: solution $v_1,\dots,v_n\in\mathbb{R}^n$ of the SDP

1. Pick a random unit vector $u\in\mathbb{R}^n$
2. Let $S_u=\{i\in V:\langle v_i|u\rangle\geq\frac{\sigma}{\sqrt{n}}\}$

   $T_u=\{i\in V:\langle v_i|u\rangle\leq\frac{-\sigma}{\sqrt{n}}\}$
3. If $|S_u|<2\cdot c'n$ or $|T_u|<2\cdot c'n$, STOP
4. while $\exists i\in S_u, j\in T_u$ s.t. $(v_i-v_j)^2\leq\Delta$,

   delete i from S_u, j from T_u
5. OUTPUT: S_u, T_u (the remaining parts)



Key Lemma: With prob. at least $\frac{6}{4} - \frac{1}{n^2}$, SET-FIND$(c', \sigma, \Delta)$ constructs sets S, T of size at least $c' \cdot n$ s.t.
$\forall i \in S,\ \forall j \in T,\quad (v_i - v_j)^2 \geq \Delta \quad \left(= \frac{\sigma^2}{4 \log n} = \Omega\left(\frac{1}{\log n}\right)\right).$

Note: much stronger claim holds: with the Δ-inequality,
$\ldots = \Omega\left(\frac{1}{\sqrt{\log n}}\right).$

Meaning: SET_FIND finds large sets far apart.

Proof - later. Usage first:

Pseudoapproximation for c-balanced cut:

1. Solve SDP
2. SET-FIND $\rightsquigarrow$ sets $S, T \subseteq V$, $|S|, |T| \geq c' \cdot n$
3. for each $e = \{i, j\} \in E$, set $w_e = (v_i - v_j)^2$, and
   for $r \in (0, \frac{\Delta}{2})$, let

"ball" around S with radius r

- $V_r = \{v \in V : dist_w(S, v) \leq r\}$ ← distance w.r.t. w
- $E_r = E(V_r, V \setminus V_r)$

wlog assume $|V_{\frac{\Delta}{2}}| \leq \frac{n}{2}$ (if not, switch S-T)

find $\bar{\alpha} = \min\limits_{r \in (0, \frac{\Delta}{2})} \frac{|E_r|}{|V_r|}$; let $\bar{r}$ be the corresponding radius in $(0, \frac{\Delta}{2})$

4. OUTPUT $(V_{\bar{r}}, V \setminus V_{\bar{r}})$

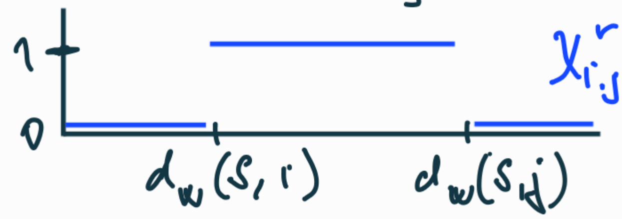
Analysis of the algorithm:

key lemma

(1) $\forall r \in (0, \frac{\Delta}{2})$: $|E_r| \geq \bar{\alpha} \cdot |V_r| \geq \bar{\alpha} \cdot |S| \geq \bar{\alpha} \cdot c' \cdot n$.

For $\{i,j\} \in E$ and $r \in (0, \frac{\Delta}{2})$, let $X^r_{i,j} = \begin{cases} 1 & \text{if } |V_r \cap \{i,j\}| = 1 \\ 0 & \text{otherwise} \end{cases}$

indicator – $\{i,j\}$ in $\delta(V_r)$?



(2) i) $\int_0^{\Delta/2} X^r_{i,j} \, dr \leq w_{i,j}$

ii) $|E_r| = \sum_{\{i,j\} \in E} X^r_{i,j} \quad \forall r \in (0, \frac{\Delta}{2})$

obvious from definitions

$$\Rightarrow \quad 4 \cdot \text{OPT-SDP} = \sum_{e \in E} w_e \overset{i)}{\geq} \sum_{e \in E} \int_0^{\Delta/2} X^r_e \, dr \overset{ii)}{=} \int_0^{\Delta/2} |E_r| \, dr \overset{(1)}{\geq}$$

$$\geq \frac{\Delta}{2} \cdot \bar{\alpha} \cdot c' \cdot n \quad , \text{ i.e., } \bar{\alpha} \leq \frac{8 \cdot \text{OPT-SDP}}{\Delta \cdot c' \cdot n}$$

$$\Rightarrow |E(V_{\bar{r}}, V \setminus V_{\bar{r}})| = \bar{\alpha} \cdot |V_{\bar{r}}| = O\left(\frac{\text{OPT\_SDP}}{\Delta \cdot n}\right) \cdot \frac{n}{2} = O(\text{OPT-SDP} \cdot \log n)$$

Theorem: There exists a randomized poly-time algorithm for c'-balanced cut of size at most $O(\text{OPT-SDP} \cdot \log n)$ (with stronger key lemma: $O(\text{OPT-SDP} \cdot \sqrt{\log n})$).

Auxiliarly Lemma: If $v \in \mathbb{R}^n$ is a vector of length ℓ, and $u \in \mathbb{R}^n$ is a randomly uniformly chosen unit vector, then i) for $x \le 1$, $\Pr\left[|\langle v,u\rangle| \le x \cdot \frac{\ell}{\sqrt{n}}\right] \le 3 \cdot x$,

ii) for $x \le \frac{\sqrt{n}}{4}$, $\Pr\left[|\langle v,u\rangle| \ge x \cdot \frac{\ell}{\sqrt{n}}\right] \le e^{-x^2/4}$.

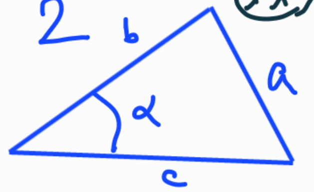
Proof postponed (skipped?)

Useful Facts (recall MAX CUT & SDP):

- for unit vectors v_i, v_j forming an angle α: (*)

$$\Pr[v_i \text{ and } v_j \text{ separated by random hyperplane}] = \frac{\alpha}{\pi}$$

- for every $\alpha \in \langle 0, \pi\rangle$: $\frac{\alpha}{\pi} \ge 0{,}878 \cdot \frac{1-\cos\alpha}{2}$ (**)
- Law of cosines: $a^2 = b^2 + c^2 - 2bc\cos\alpha$



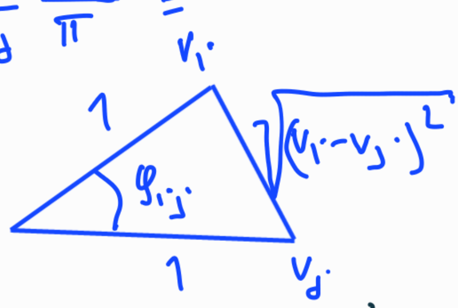
Proof of Key Lemma:

Think about the random hyperplane with normal vector u chosen in the first step of SET-FIND procedure (by φ_{ij} we denote the angle of v_i and v_j):

$$\mathbb{E}[\#\text{ pairs of vectors } v_i, v_j \text{ separated by } u] \overset{(*)}{=} \sum_{i<j} \frac{\varphi_{ij}}{\pi} \overset{(**)}{\ge}$$

$$\ge \sum_{i<j} 0{,}878 \cdot \frac{1-\cos\varphi_{ij}}{2} \underset{\text{Law cosines}}{=} 0{,}878 \cdot \sum_{i<j} \frac{(v_i - v_j)^2}{4} \underset{\text{SDP}}{\ge}$$

$$\ge \overbrace{0{,}878 \cdot c \cdot (1-c)}^{=b} \cdot n^2 = b \cdot n^2$$



$$\cos\varphi_{ij} = \frac{2 - (v_i - v_j)^2}{2}$$

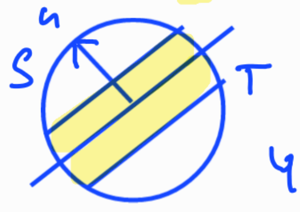
$\Rightarrow \mathbb{E}[\#\text{ non-separated pairs} \ldots] \le n^2 - bn^2$

By Markov inequality:

$$\Pr\left[\#\text{ non-separated pairs} \ldots \ge n^2 - \frac{bn^2}{2}\right] \le \frac{n^2 - bn^2}{n^2 - \frac{bn^2}{2}} = \frac{1-b}{1-\frac{b}{2}} \le 1 - \frac{b}{2}$$

Thus, with prob. at least $\frac{b}{2}$, # of separ. vectors is at least $\frac{bn^2}{2}$

$\Rightarrow$ in the smaller part is $\ge \frac{bn}{2}$ vectors.



By Auxiliarly Lemma, part i):

$$\mathbb{E}\left[\#\text{ vectors in equatorial zone of width } \frac{\sigma}{\sqrt{n}}\right] \le 3\sigma n.$$

By Markov inequality and $\sigma = \frac{b^2}{48}$:

$$\Pr\left[\#\text{ vectors in equatorial} \dots \ge \frac{bn}{4}\right] \le \frac{3\sigma n}{\frac{bn}{4}} =$$

$$= \frac{3\cdot b^2 \cdot n \cdot 4}{48 \cdot b \cdot n} = \frac{b}{4}$$

$\Rightarrow$ with prob. $\ge \frac{b}{2} - \frac{b}{4} = \frac{b}{4}$, the sizes of S_u, T_u at the end of step 2 are at least $\frac{b\cdot n}{2} - \frac{b\cdot n}{4} = \frac{bn}{4} = 2\cdot c'\cdot n$

(for $c' = \frac{b}{8} = 0{,}878 \cdot \frac{c\cdot(1-c)}{8}$).

Consider a pair of vectors v_i, v_j deleted in step 4:

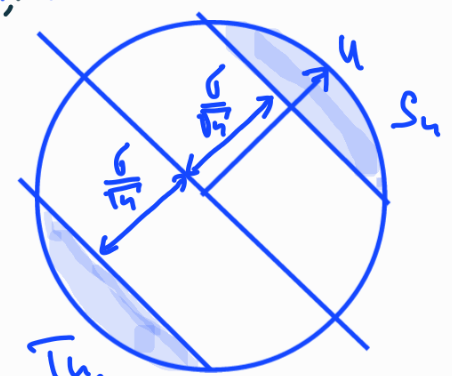
- $(v_i - v_j)^2 \le \Delta = \frac{\sigma^2}{4\log n}$ as they are deleted

Note: $v_i - v_j$ is small with large projection

(∗∗)

- $|\langle (v_i - v_j) | u \rangle| \ge \frac{2\sigma}{\sqrt{n}} = \overbrace{4\sqrt{\log n}}^{\text{"x"}} \overbrace{\frac{\sigma}{2\sqrt{\log n}\sqrt{n}}}^{\text{"ℓ"}}$

as one of them in S_u, the other in T_u



By Auxiliarly Lemma, part ii):

$$\Pr[(**)] \le e^{-\frac{16\log n}{4}} < \frac{1}{n^4}$$

$$\Rightarrow \mathbb{E}[\#\text{ pairs deleted in step 4}] < n^2 \cdot \frac{1}{n^4} = \frac{1}{n^2}$$

By Markov inequality:

otherwise no pair deleted

$$\Pr[\#\text{ pairs deleted in step 4} \ge 1] < \frac{1}{n^2}$$

Putting everything together:

$\Rightarrow$ with prob. $\ge \frac{b}{4} - \frac{1}{n^2}$, $|S_u|, |T_u| \ge 2\cdot c'\cdot n$.