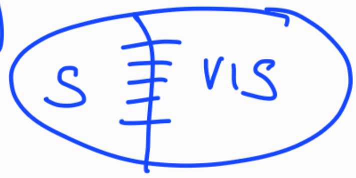
SPANNING TREE CONGESTION

12/12/2025

Terminology, notation: cut of $G=(V,E)$ - subset of vertices $S \subseteq V$

(or: a partition of V into S and $V \setminus S$)



cut size $\quad e_G(S, V\setminus S) = |E_G(S, V\setminus S)|$

For a tree T: every edge uv induces a cut -

- the two connected components of $T \setminus uv$

Congestion of the edge $uv \in T$ w.r.t. G and its spanning tree T: $\quad c_{G,T}(uv) = e(S_u, S_v)$

Congestion of the tree T

$$C(G,T) = \max_{uv \in T} c_{G,T}(uv)$$

spanning tree congestion of G (1987, Simonson, different name)

$$STC(G) = \min_{T\text{ - span. tree of } G} C(G,T)$$

PROBLEMS:

- STC : what is $STC(G)$?
- k-STC - is $STC(G) \le k$?

Known: • NP-hard, 4-STC NP-hard [2025] $\Rightarrow$ no $c < \frac{5}{4}$-approx, unless P=NP

3-STC poly [2010]

• $\frac{n}{2}$-approximation – any spanning tree

n-approx: lower bound: $\frac{m}{n-1}$

upper b.: any spanning tree-coloring: $\le m$

$\Rightarrow \frac{m}{\frac{m}{n-1}} \le n$

$\frac{n}{2}$-approx:

better lower bound: $L = \frac{n-1+2(m-n+1)}{n-1}$ m−n+1 edges mapped to paths of length ≥ 2

any spanning tree: $1 + m-(n-1) \le$

$\frac{n}{2} + m - n + 1 \le \frac{n}{2}\left(1 + \frac{2(m-n+1)}{n-1}\right) = \frac{n}{2} \cdot L$ $\blacksquare$

• OPEN PROBLEM: $o(n)$-approximation

• for fixed k, on bounded treewidth graphs – Poly

– on bounded degree – Poly

• exact $O(2^n)$-time alg.

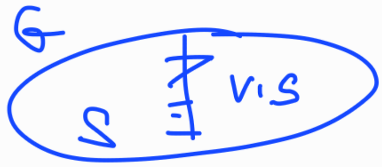
• $O(\Delta \log^{3/2} n)$-approximation for graphs of max degree Δ

[K. 2025]

TOOLS:

c-balanced cut of G ($c \geq \frac{1}{2}$) - partition of V into S and $V \setminus S$ s.t. $|S|, |V \setminus S| \leq c \cdot n$

special case: $c = \frac{1}{2}$. **bisection** of $G = (V, E)$



$$b(G) = \min_{\substack{S \subseteq V, \\ |S| = \frac{n}{2}}} e(S, V \setminus S)$$ **min. bisection**

$$hb(G) = \max_{\substack{H \text{ subgraph} \\ \text{of } G}} b(G)$$ **hereditary bisection**

$b(G) = 0 \quad hb(G) = \left(\frac{n}{4}\right)^2$

i.e., $hb(G) \gg b(G)$

THM 1 (Arora, Rao, Vazirani, 2004): A $2/3$-balanced cut of G of cost $O(\sqrt{\log n} \cdot b(G))$ can be computed in poly time.

edge expansion of $G = (V, E)$

$$\beta(G) = \min_{A \subseteq V} \frac{e(A, V \setminus A)}{\min\{|A|, |V \setminus A|\}}$$

$\beta(G) \leq \frac{b(G)}{n/2}$ by definition

THM 2 (K., Matoušek, 2004): Every graph G contains a subgraph H on at least $\frac{2n}{3}$ vertices s.t.

$$\beta(H) \geq \frac{b(G)}{n}.$$

LOWER BOUNDS II – NEW

THM 3: $$STC(G) \geq \frac{2 \cdot hb(G)}{3 \cdot \Delta}$$

Lemma: For every subgraph H of G:
$$STC(G) \geq \frac{\beta(H) \cdot n'}{\Delta} \quad \text{where } n' = |V(H)|.$$

Proof of THM 3: Lemma $\Rightarrow$ THM 3:
Consider the subgraph H of G of max. bisection, i.e.
$$b(H) = hb(G).$$

By THM 2, there is a subgraph H' of H
s.t. $\beta(H') \geq \frac{b(H)}{n'}$ and $|V(H')| \geq \frac{2}{3} n'$:

Apply Lemma to H': $\frac{\beta(H') \cdot |V(H')|}{\Delta} \geq \frac{b(H)}{n'} \cdot \frac{2}{3} \cdot \frac{n'}{\Delta} = \frac{2hb(G)}{3\Delta}$. $\blacksquare$

Given a tree T on n vertices with $n' \leq n$ vertices marked, there $\exists$ a vertex whose removal splits T into components each with $\leq \frac{n'}{2}$ marked vertices.

Proof of Lemma:
Let T be opt. spanning tree of G, w.r.t. STC.
by 👁, $\exists\, z \in T$ s.t. each component of $T \setminus z$ contains $\leq \frac{n'}{2}$ vertices from H'

as max degree of T is Δ, at least one of the components, say C, contains $\geq \frac{n'}{\Delta}$ vertices from H

$$\Rightarrow e(C, V \setminus C) \geq e(C \cap V(H), V(H) \setminus C) \underset{\text{expansion}}{\geq} \beta(H) \cdot \frac{n'}{\Delta}$$

Every edge between C and $V \setminus C$ in G is mapped to the single edge between C and $V \setminus C$ in T

$$\Rightarrow STC(G) \geq \frac{\beta(H) \cdot n'}{\Delta} \quad \blacksquare$$

APPROXIMATION ALGORITHM Cong ST(H)

1 **if** $|V(H)| = 1$ **then return** H

2 construct $\frac{2}{3}$-balanced cut $(S, V \setminus S)$ by ARV-alg.

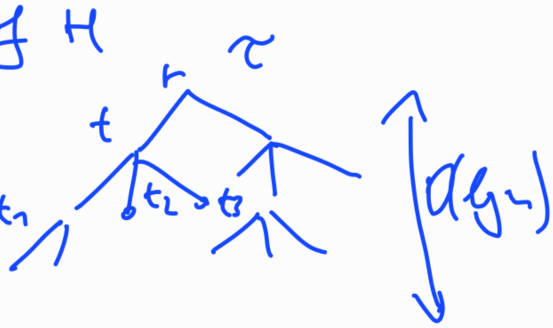
3 $F = E(S, V \setminus S)$ $\qquad |F| \le O(\sqrt{\log n}) \cdot b(H)$

4 **for each** component C of $H \setminus F$ **do**

$\qquad T_C = \text{Cong ST}(C)$

5 arbitrarily connect the trees T_C by edges from F into a spanning tree T of H

6 **return** T



τ – tree representing the recursive decomposition of G:

root r corresponds to $G = (V, E)$

children of $t \in \tau$ … connected components of $G[V_t] \setminus F$, where F is the $\frac{2}{3}$-balanced cut of $G[V_t]$ - step 4

Lemma: For $t \in \tau$ with children $t_1, .., t_k$:

$$C(G_t, T_t) \le \max_i C(G_{t_i}, T_{t_i}) + O(\sqrt{\log n}) \cdot b(G_t)$$

Proof: For $e \in T_t \cap T_{t_i}$, for some i: $c_{G_t, T_t}(e) \le C(G_{t_i}, T_{t_i}) + |F|$.

For $e \in T_t \cap F$: $c_{G_t, T_t}(e) \le |F|$. ∎

Corollary: $C(G, T) = O(\log^{3/2} n) \cdot hb(G)$

Proof: by induction on the height $h(t)$ of a subtree V_t:

$C(G_t, T_t) \le h(t) \cdot O(\sqrt{\log n}) \cdot hb(G)$ at every level increase by $\le O(\sqrt{\log n}) \cdot hb(G)$.

THM 4: The approx. ratio of Cong ST is $O(\Delta \cdot \log^{3/2} n)$.

Proof: THM 2 (lower bound) + Corollary. ∎

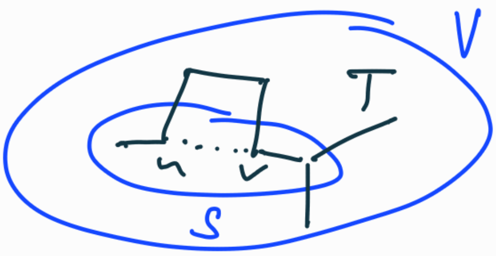
Another lower bound:

Lemma: For every $G=(V,E)$ and every connected $S \subseteq V$,

$$STC(G) \geq \frac{STC(G[S])}{e(S, V \setminus S)}$$

Proof: Let T be an optimal spanning tree of G.

Consider $S \subseteq V$. Let $F \subseteq E \cap (S \times S)$ be edges s.t. the u-v-path in T uses $w \notin S$



- for every $uv \in F$, the u-v-path in T uses ≥ 2 edges from $E(S, V \setminus S)$

$$\Rightarrow |F| \leq STC(G) \cdot \frac{e(S, V \setminus S)}{2}$$

Let T' be an arbitrary extension of $T[S]$ into a spanning tree of $G[S]$. Then

$$c(G[S], T') \leq STC(G) + |F| \leq STC(G)\left(1 + \frac{e(S, V \setminus S)}{2}\right).$$

If $e(S, V \setminus S) = 1$: $STC(G) \geq STC(G[S]) = \dfrac{STC(G[S])}{e(S, V \setminus S)}$ ✓

If $e(S, V \setminus S) \geq 2$: $STC(G) \geq \dfrac{c(G[S], T')}{1 + \frac{e(S, V \setminus S)}{2}} \geq \dfrac{STC(G[S])}{e(S, V \setminus S)}$ ✓

$\blacksquare$

Ex. 3-regular expander graph G': $STC(G') \overset{\text{known}}{=} \Omega(n)$

let $G =$ 

$\Delta(G) = n$

$STC(G) = O(1)$

opt. tree is star

The existence of a subgraph of a large STC does NOT imply large STC of the entire graph. !