INTERIOR POINT METHODS – part 2   21/12/2020

PRIMAL-DUAL AFFINE SCALING POTENTIAL REDUCTION ALGORITHM

RECAP: $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$

(P) $\min c^T x$ s.t. $Ax = b$, $x \geq 0$ dual (D) $\max y^T b$ s.t. $A^T y + s = c$, $s \geq 0$

Duality gap: $c^T x - y^T b = s^T x$ $L = \langle A, b \rangle$ … input size

We observed: if $x^T s \leq 2^{-2L}$ then any vertex x' s.t. $c^T x' \leq c^T x$ is an optimal solution.

$\Rightarrow$ our goal is to find a pair of primal-dual solutions x, s s.t. $x^T s \leq 2^{-2L}$.

TOOLS

- Potential function: for $x > 0$, $s > 0$

$$G(x,s) = (n+\sqrt{n}) \ln x^T s - \sum_{i=1}^{n} \ln (x_i s_i)$$

We proved:

If $G(x,s) \leq -2\sqrt{n}\,L$, then $x^T s < e^{-2L}$.

i.e. to meet our goal, it suffices to find strictly feasible x, s with "small" potential

- Scaling: for $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n) > 0$, let $\bar{X} = \mathrm{diag}(\bar{x}_1, \dots, \bar{x}_n)$

Then an affine mapping given by $x' = \bar{X}^{-1} \cdot x$ moves $\bar{x}$ to $e = (1, \dots, 1)$ that is far from boundaries

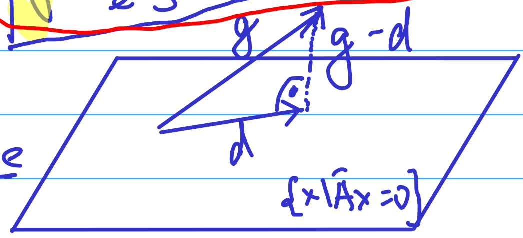
$\rightsquigarrow$ transformed variables x', s':

(P') $\min \bar{c}^T x'$, $\bar{A} x' = b$, $x' \geq 0$ (D') $\max b^T y$, $\bar{A}^T y + s' = \bar{c}^T$, $s' \geq 0$ where $\bar{c}^T = c^T \bar{X}$, $\bar{A} = A \cdot \bar{X}$

it holds: $x^T s = x'^T s'$ i.e., the same duality gap!

- For a feasible solution $e = (1, \dots, 1)$ of (P') and s' of (D') the gradient of $G(x,s)$ at (e, s') is $\boxed{g = \frac{n+\sqrt{n}}{e^T s'} s' - e}$

let d be the projection of g to $\{\bar{A}x = 0\}$



Play: change the primal feasible solution e in the direction $-d$

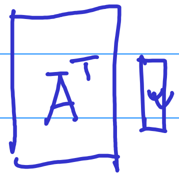
Our assumption: A has a full row rank ; $\Rightarrow \bar{A}$ ... as well
$\Rightarrow \bar{A}\bar{A}^T$ is regular !

Lemma: $$d = \left(I - \bar{A}^T(\bar{A}\bar{A}^T)^{-1}\bar{A}\right)g$$

Recall Fact 2: for x, a s.t. $|x| \le a < 1$: $-x - \frac{x^2}{2(1-a)} \le \ln(1-x) \le -x$

Proof of Lemma:

- as $g-d$ is perpendicular to $\{x \mid \bar{A}x = 0\}$, $\exists\, w$ s.t. $\bar{A}^T w = g - d$.
- as $d \in \{x \mid \bar{A}x = 0\}$: $\bar{A}d = 0$



$$(\bar{A}\bar{A}^T)\,w = \underbrace{\bar{A}(g-d)}_{\bar{A}g}$$

$$w = (\bar{A}\bar{A}^T)^{-1}\cdot\bar{A}(g-d)$$

$$d = g - \bar{A}^T(\bar{A}\bar{A}^T)^{-1}\cdot\bar{A}\,g \quad \blacksquare$$

PRIMAL STEP

$$\tilde{x}_j := 1 - \frac{1}{4\|d\|}d_j$$

if $\|d\| \ge 0{,}2$ then: $\boxed{\tilde{x} = e - \frac{1}{4\|d\|}d, \quad \tilde{s} = s'}$

Lemma: $\tilde{x} > 0$ (i.e., $\tilde{x}$ is again strictly feasible)

Proof! $\frac{d_j}{\|d\|} \le 1 \quad \forall j \qquad \tilde{x}_j = 1 - \frac{1}{4}\frac{d_j}{\|d\|} \ge \frac{3}{4} > 0$ □

Lemma: $$\underbrace{G(\tilde{x},\tilde{s}) - G(e,s')}_{(*)} \le \frac{-1}{120}$$

Proof: let $q = n + \sqrt{n}$. Then $G(x,s) = q\cdot\ln x^Ts - \sum_{j=1}^{n}\underbrace{\ln x_j s_j}_{\ln x_j + \ln s_j}$

$$(*) = q\cdot\ln\left(e^Ts' - \frac{d^Ts'}{4\|d\|}\right) - \sum_{j=1}^{n}\ln\left(1 - \frac{d_j}{4\|d\|}\right) - \sum_{j=1}^{n}\ln s'_j$$

$$-q\ln e^Ts' + \sum_{j=1}^{n}\ln s'_j$$

$$= q\ln\left(1 - \frac{d^Ts'}{4\|d\|e^Ts'}\right) - \sum_{j=1}^{n}\ln\left(1 - \underbrace{\frac{d_j}{4\|d\|}}_{\le \frac{1}{4} = a}\right)$$

bounds on log.

$$-x - \frac{x^2}{2(1-a)} \le \ln(1-x) \le -x \quad \text{if } |x| \le a < 1$$

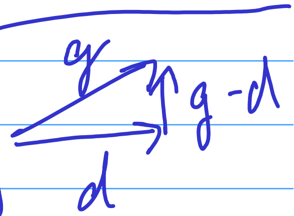
$$\le -\frac{g\, d^T s'}{4\|d\|\, e^T s'} + \sum_{j=1}^{n} \frac{d_j}{4\|d\|} + \sum_{j=1}^{n} \frac{d_j^2}{16\|d\|^2 \cdot 2\left(1-\frac{1}{4}\right)}$$

$$\le -g \frac{d^T s'}{4\|d\|\, e^T s'} + \frac{d^T e}{4\|d\|} + \frac{1}{24}$$

$$= \frac{d^T}{4\|d\|}\Big(\underbrace{e - g\frac{s'}{e^T s'}}_{=-g}\Big) + \frac{1}{24} = -\frac{d^T g}{4\|d\|} + \frac{1}{24}$$

$\|d\| \geq 0{,}2$

$$= -\frac{\|d\|^{\cancel{2}}}{4\cancel{\|d\|}} + \frac{1}{24} = \frac{-\|d\|}{4} + \frac{1}{24} \le \frac{-0{,}2}{4} + \frac{1}{24} = \frac{-1}{120}$$



$(g-d)^T d = 0$

$g^T d = d^T d$

DUAL STEP

gradient of $G(x,s)$ with respect to dual variables s evaluated at (e,s')

$$h = \nabla_s G(x,s)\Big|_{(e,s')} = (n+\sqrt{n})\frac{e}{e^T s'} - \begin{pmatrix} \frac{1}{s'_1} \\ \vdots \\ \frac{1}{s'_n} \end{pmatrix}$$

$$\left[\; g = (n+\sqrt{n})\frac{s'}{e^T s'} - \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \;\right]$$

Note: $h_j = \frac{g_j}{s'_j}$ $\qquad s' > 0$

$\Rightarrow$ g and h are in the same ortant

$\Rightarrow$ we will work with g instead of h

$\tilde s$ $\quad$ (s', y') ... current dual solution

Idea: to change s' in the direction $-g$

BUT we have to ensure:

$$\exists y \text{ s.t. } \bar A^T y + \tilde s = \bar c$$

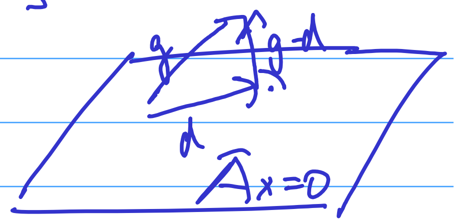
We know: $\bar A^T y' + s' = \bar c$

$\Rightarrow$ it has to hold: $\tilde s - s' = \bar A^T (y' - y)$

i.e. $\tilde s - s'$ is a linear combination of rows of $\bar A$

i.e. $\tilde s - s'$ is perpendicular to $\{\bar A x = 0\}$

$\rightsquigarrow$ we know that $(g-d) \perp \{\bar A \cdot x = 0\}$



idea: change s' in the direction $-(g-d)$

$$\boxed{\tilde s = s' - (g-d)\mu}$$

1: $\forall \mu \ \exists y$ s.t. $\bar A^T \cdot y + \tilde s = \bar c$

Proof: has $\bar A^T \cdot y + s' - (g-d)\mu = \bar c$ a solution?

we know: $s' = \bar c - \bar A^T \cdot y'$

has $\bar A^T (y - y') = (g-d)\mu$ a solution?

YES, as $(g-d)$ is by definition a lin. comb. of rows of $\bar A$. ∎

DUAL STEP (i.e., if $\|d\| < 0.2$): set $\mu = \frac{e^T s'}{n + \sqrt n}$

2: Then $\tilde s = s' - (g-d)\frac{e^T s'}{n+\sqrt n} > 0$ $\qquad |d_j| < 0.2$

Proof: $\boxed{\tilde s} = \cancel{s'} - \frac{e^T s'}{n+\sqrt n}\Big(\underbrace{\frac{n+\sqrt n}{\cancel{e^T s'}} s' - e}_{= g} - d\Big) = \boxed{\frac{e^T s'}{n+\sqrt n}(d+e)} > 0$ ∎

Lemma: $\tilde{s}$ is a dual strictly feasible solution.
by ⚙ 1 & 2

Goal: show that potential decreases "a lot"
in dual step.

Lemma AUX: $\sum_{j=1}^{n} \ln \tilde{s}_j \geq n \cdot \ln \frac{e^T \tilde{s}}{n} - \frac{1}{40}$.

Note: $e^T \tilde{s} = \frac{e^T s'}{n+\sqrt{n}} (e^T d + n) \leq \frac{e^T s'}{n+\sqrt{n}} (\sqrt{n} \cdot \|d\| + n) \leq$ $\|d\| \leq 0{,}2$

$\tilde{s} = \frac{e^T s'}{n+\sqrt{n}} (d+e)$

Cauchy-Schwarz

$\leq \frac{e^T s'}{n+\sqrt{n}} \left(n + 0{,}2\sqrt{n}\right)$

$\Rightarrow$ ⚙ 3: $\frac{e^T \tilde{s}}{e^T s'} \leq \frac{n + 0{,}2\sqrt{n}}{n+\sqrt{n}}$

Lemma: In the dual step:
$G(\tilde{x}, \tilde{s}) - G(e, s') \leq - \frac{3}{\rho}$.