INTERIOR POINT METHODS – part 1

14/12/2020

PRIMAL-DUAL AFFINE SCALING POTENTIAL REDUCTION ALGORITHM

PRELIMINARIES

Given: integer matrix $A^{m\times n} \in \mathbb{Z}$, integer vectors $b \in \mathbb{Z}^m$, $c \in \mathbb{Z}^n$

Goal: to find an optimal solution

$$\min c^T x \quad \text{s.t. } Ax = b,\ x \geq 0 \qquad (P)$$

let $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$

(D) $\max y^T b$
$A^T y + s = c$
$s \geq 0$ $\quad$ s … slack variables

$L = \langle A \mid b \rangle$ input size

Assume, wlog: $\operatorname{rank}(A) = m$ (Then AA^T is regular)

$\Rightarrow$ for a feasible (y, s) of (D), y is uniquely given by s

Duality gap:

$$c^T x - y^T b = y^T A x + s^T x - y^T A x = s^T x$$

Theorem 1: Under the above assumptions, let u and v be vertices of P. If $c^T u \neq c^T v$, then

$$|c^T u - c^T v| > 2^{-2L}.$$

Proof: by the same arguments* as in the proof of the last lemma in the first lecture on ellipsoid alg. we know:

- $u_i = \dfrac{\det\ldots}{\det(\cdot)}$, $\forall i$ $\quad$ $\det(\cdot) = q$ … submatrix of $\binom{A}{I}$
- $v_i = \dfrac{\det\ldots}{\det(\cdot)}$ $\quad$ $\det(\cdot) = p$ … also a submatrix of $\binom{A}{I}$

we also know:

- $|\det B| < 2^L$ for any square submatrix of A

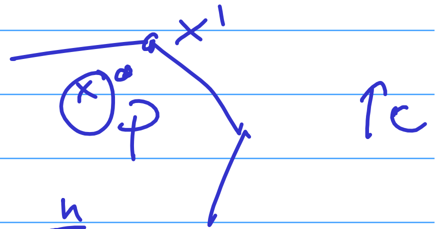
$$\Rightarrow \underbrace{|c^T u - c^T v|}_{\neq 0} = \left| \frac{q \cdot c^T u}{q} - \frac{p \cdot c^T v}{p} \right| = \left| \frac{q \cdot p\,(c^T u - c^T v)}{2 \cdot p} \right| \geq \frac{1}{4 \cdot 2} \geq \frac{1}{2^{2L}}$$

integer!

$\square$

* see the last page of lecture notes for this lecture

Corollary: If $x^Ts \le 2^{-2L}$, for feasible solutions x of (P) and s of (D), then any vertex x' s.t. $c^Tx' \le c^Tx$ is an optimal solution.



POTENTIAL FUNCTION

Def: $G(x,s) = (n+\sqrt{n}) \ln x^Ts - \sum_{i=1}^{n} \ln(x_i s_i)$ for $x>0$, $s>0$

Fact 1: For any $t_j \ge 0$, $j=1,\dots,n$:

geometric and arithmetic means

$$\left(\prod_{j=1}^{n} t_j\right)^{1/n} \le \frac{1}{n}\sum_{j=1}^{n} t_j \qquad (*)$$

Fact 2: For x, a s.t. $|x| \le a < 1$:

$$-x - \frac{x^2}{2(1-a)} \le \ln(1-x) \le -x$$

Lemma: If $G(x,s) \le -2\sqrt{n}\,L$, then $x^Ts < e^{-2L}$.

Proof: For $j=1,\dots,n$, let $t_j = x_j s_j$.

Consider ln of $(*)$:

$$\frac{1}{n}\ln \prod_{j=1}^{n} x_j s_j \le \ln\left(\frac{1}{n}x^Ts\right)$$

$$\frac{1}{n}\sum_{j=1}^{n} \ln x_j s_j \le \ln x^Ts - \ln n \qquad (**)$$

scale by n

Thus:

$$-2\sqrt{n}\,L \overset{\text{our assumption}}{\ge} G(x,s) \overset{\text{definition of } G}{=} (n+\sqrt{n})\ln x^Ts - \sum_{j=1}^{n} \ln x_j s_j \ge$$

$$\ge \sqrt{n}\ln x^Ts + \underbrace{n \ln n}_{\ge 0} \ge \sqrt{n} \cdot \ln x^Ts$$

$$\Rightarrow \quad e^{-2L} \ge e^{\ln x^Ts} = x^Ts$$

∎

Assume that an optimal solution exists

Algorithm - sketch

1. Find a strictly feasible solutions $x_0 > 0$, $s_0 > 0$ s.t. $G(x_0, s_0) = O(\sqrt{n} L)$; $k = 0$
2. While $x_k^T s_k \geq e^{-2L}$,
   Find strictly feasible solutions $x_{k+1} > 0$, $s_{k+1} > 0$ s.t. $G(x_k, s_k) - G(x_{k+1}, s_{k+1}) > \varepsilon$
   $k = k+1$
3. "Jump" from x_k to an optimal solution.

SCALING

Assume we have a strictly feasible $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n) > 0$ and a strictly feasible $\bar{s} = (\bar{s}_1, \dots, \bar{s}_n) > 0$.

we want a mapping : $\bar{x} \to (1, 1, \dots, 1)$

let $\bar{X} = \begin{pmatrix} \bar{x}_1 & & & 0 \\ & \bar{x}_2 & & \\ & & \ddots & \\ 0 & & & \bar{x}_n \end{pmatrix}$ $\qquad \bar{X}^T = \bar{X}$

Then $\bar{X}^{-1} \cdot \bar{x} = (1, 1, \dots, 1)$ ... i.e., the desired mapping

in general : $\underline{x' = \bar{X}^{-1} x}$ $\qquad x = \bar{X} x'$

rewrite (P) in terms of transformed variables :

$$\min \; c^T \bar{X} x' \qquad \text{let } \bar{c}^T = c^T \bar{X}$$

$$A \bar{X} x' = b \qquad \bar{A} = A \bar{X}$$

$$x' \geq 0$$

(P') $\min \; \bar{c}^T \cdot x'$ $\qquad$ x is feasible for (P)

$\to \bar{A} x' = b$ $\qquad$ $\Leftrightarrow x' = \bar{X}^{-1} x$ is feasible for (P')

$x' \geq 0$

Dual (D') $\max \; b^T y$

$\to (A \bar{X})^T y + s' = c^T \bar{X}$ $\quad \rightsquigarrow \; s' = \bar{X} s$

$s' \geq 0$

$$\boxed{\text{Recall } G(x,s) = (n+\sqrt{n})\ln x^T s - \sum_{j=1}^{n} \ln x_j s_j}$$

$$s' = \begin{pmatrix} s_1 \bar{x}_1 \\ s_2 \bar{x}_2 \\ \vdots \\ s_n \bar{x}_n \end{pmatrix}$$

$$\underbrace{x^T s} = x^T \underbrace{\bar{X}^T \bar{X}^{-1}}_{I} \cdot s' = \underbrace{x'^T \cdot s'}$$

i.e. the duality gap remains the same.

HOW TO DO THE ITERATIVE STEP

given a pair of feasible solutions • $\overbrace{(1,1,\dots,1)}^{=e}$ .. of (P)

• s' ... of (D)

gradient of the potential function

let $g = \nabla_x G(x,s)\Big|_{(e,s')} = \underbrace{\frac{n+\sqrt{n}}{e^T s'}\cdot s' - e}$

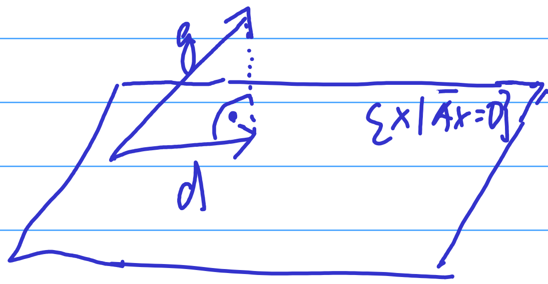
partial derivatives

$$\frac{\partial}{\partial x_i} G(x,s)\Big|_{(e,s')} = \left[\frac{(n+\sqrt{n})}{x^T s}\cdot s_i - \frac{s_i}{x_i s_i}\right]_{(e,s')} = \underbrace{\frac{n+\sqrt{n}}{e^T s} s_i' - 1}$$

First idea: change e in the direction $-g$.

Note: by doing so, we might violate the constraints $Ax=b$

let d be the projection of g to the $\{x \mid \bar{A}x = 0\}$



Second idea: change e in the direction $-d$

(i.e., $e \rightsquigarrow e + \alpha\cdot(-d)$, for some $\alpha>0$)

Lemma: $d = (I - \bar{A}^T(\bar{A}\bar{A}^T)^{-1}\bar{A})\cdot g$

An addendum for the proof of Thm 1:

Theorem: let F be a minimal nonempty face of $P=\{x \mid Cx \le d\}$.
Then $F=\{x \mid C'x = d'\}$ for some subsystem $C'x \le d'$ of $Cx \le d$.

Note: a minimal nonempty face of $P = \{x \mid Ax=b,\ x\ge 0\}$ is a vertex.

Thus, for every vertex v of P there is a system of linear equations - derived from $Ax=b,\ x\ge 0$, s.t. v is the unique solution

$\Rightarrow$ v can be expressed by Cramer's rule.