ELLIPSOID ALGORITHM - PART 3

7/12/2020

Goal: find $x \in \boxed{P = \{x \in \mathbb{R}^n \mid Cx \le d\}}$

Assumptions:
- P is bounded, i.e., $\exists K > 0$ s.t. $P \subseteq \{x \in \mathbb{R}^n \mid -K \le x_i \le K,\ i = 1, \dots, n\}$
- if "non-empty", then P has full dimension

1. $E_0 = E(z_0, C_0)$ for $R = \sqrt{n} \cdot 2^{\langle C, d\rangle - n^2}$, $C_0 = R \cdot I$, $z_0 = 0$, $k = 0$
2. While $z_k \notin P$ (violated inequality: $c^T x \le \dots$)

$$E_{k+1} = E(z_{k+1}, C_{k+1}) \text{ for } z_{k+1} = z_k - \frac{1}{n+1} \frac{C_k c}{\sqrt{c^T C_k c}}$$

$$C_{k+1} = \frac{n^2}{n^2 - 1}\left(C_k - \frac{2}{n+1} \cdot \frac{C_k^T c c^T C_k}{\sqrt{c^T C_k c}}\right)$$

$k = k + 1$

if k "too large", output "P is empty", STOP

3. OUTPUT z_k

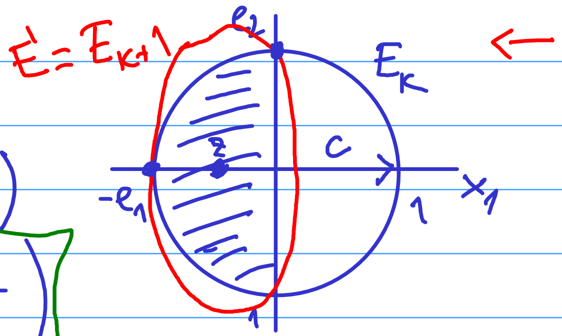
Invariant: $P \subseteq E_i$ in every iteration

We can handle the simple case:

$E_k = E(0, I)$, $c = e_1 = (1, 0, \dots, 0)$

$\Rightarrow E_{k+1} = E(z, Z)$ for $z = \left(\frac{-1}{n+1}, 0, \dots, 0\right)$

$$Z = \boxed{\mathrm{diag}\left(\frac{n^2}{(n+1)^2}, \frac{n^2}{n^2 - 1}, \dots, \frac{n^2}{n^2 - 1}\right)}$$



We know: $\mathrm{Vol}(E_{k+1}) \le e^{-\frac{1}{2(n+1)}} \mathrm{Vol}(E_k)$

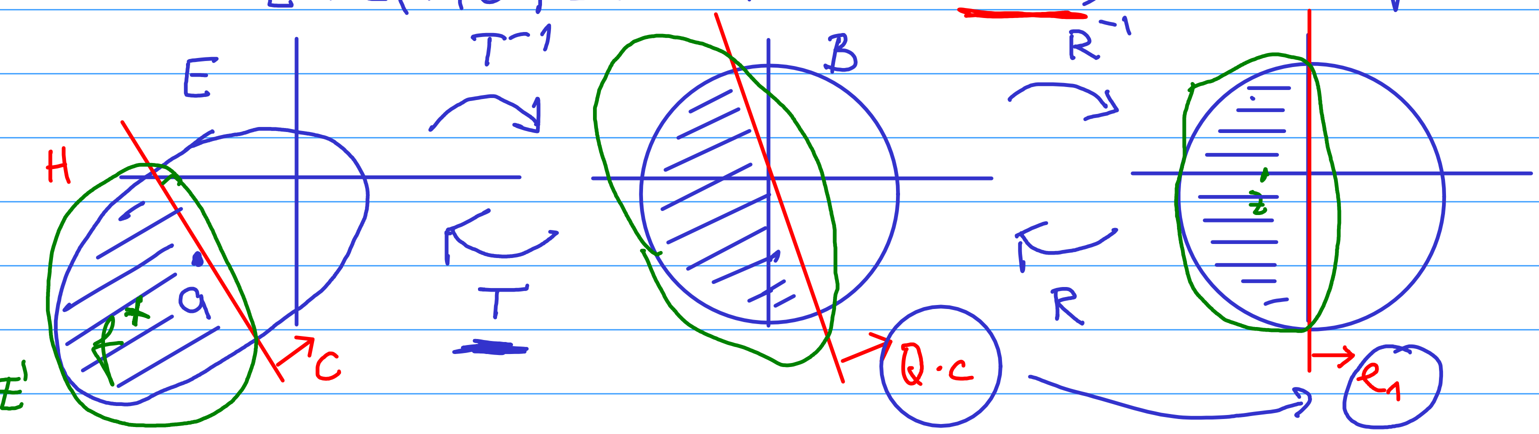
General case

Given: $E = E(a, A) = Q \cdot E(0, I) + a$ where $Q = A^{1/2}$

$H = \{x \in \mathbb{R}^n \mid c^T x \le c^T a\} = \{x \in \mathbb{R}^n \mid c^T(x - a) \le 0\}$

Looking for

$$E' = E(f, C) \supseteq E \cap H \text{ with } \mathrm{vol}(E') \text{ as small as possible}$$



We know: $T(y) = Q\cdot y + a$ $\qquad T^{-1}(x) = Q^{-1}(x-a)$

$H \cap E \subseteq T\left(R\left(E\left(\frac{-e_1}{n+1}, Z\right)\right)\right)$ $\qquad R(e_1) = \frac{Qc}{\|Qc\|}$

$\rightarrow$ $f = a - \frac{1}{n+1}\cdot\frac{Ac}{\sqrt{c^T A c}}$ center of new ellipsoid E'

$\rightarrow$ $x - f = x - T(R(z)) = x - a - QRz = QR\left(R^{-1}Q^{-1}(x-a) - z\right)$

Fact: For rotation matrix R: $R^T R = I$; i.e., $R^{-1} = R^T$

Recall that $(Xu)^T = u^T X^T$, $\quad (XYV)^{-1} = V^{-1}Y^{-1}X^{-1}$

Proof of the Half ellipsoid lemma from previous lecture

$E' = E(f, C) = \{\underbrace{T(Ry)}_{=x} \mid (y-z)^T Z^{-1}(y-z) \leq 1\} =$ $\qquad$ subst. $x = TR(y)$ i.e. $y = R^{-1}T^{-1}(x)$

$= \{x \in \mathbb{R}^n \mid (R^{-1}T^{-1}(x) - z)^T Z^{-1}(R^{-1}T^{-1}(x) - z) \leq 1\}$

by def of T^{-1} $\quad = \{x \in \mathbb{R}^n \mid (R^{-1}Q^{-1}(x-a) - z)^T Z^{-1}(R^{-1}Q^{-1}(x-a) - z) \leq 1\}$

by 💡 $\quad = \{x \in \mathbb{R}^n \mid (R^{-1}Q^{-1}(x-f))^T Z^{-1}(R^{-1}Q^{-1}(x-f)) \leq 1\}$

$= \{x \in \mathbb{R}^n \mid (x-f)^T Q^{-1^T} R^{-1^T} Z^{-1} R^{-1} Q^{-1}(x-f) \leq 1\}$

"C^{-1}"

we know

$C = Q\cdot R\cdot Z\cdot R^T\cdot Q^T$

$$= \frac{n^2}{n^2-1}\cdot Q\cdot R\cdot \underbrace{\begin{pmatrix} \frac{n-1}{n+1} & & & 0 \\ & 1 & & \\ & & 1 & \\ & & & \ddots \\ 0 & & & & 1 \end{pmatrix}}_{= I - \begin{pmatrix} \frac{2}{n+1} & \cdots & 0 \\ \vdots & 0 & \\ 0 & & \ddots & 0 \end{pmatrix}} \cdot R^T\cdot Q^T$$

$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \\ 0 & & & 0 \end{pmatrix} = e_1 e_1^T$

$$= \frac{n^2}{n^2-1}\left(\underbrace{Q\cdot R\cdot I\cdot R^T\cdot Q^T} - \frac{2}{n+1} Q\cdot R\cdot e_1\cdot e_1^T\cdot R^T\cdot Q^T\right)$$

$$= \frac{n^2}{n^2-1}\left(A - \frac{2}{n+1}\,\frac{A\cdot c}{\|Qc\|}\cdot\frac{c^T A^T}{\|Qc\|}\right)$$

$$= \frac{n^2}{n^2-1}\left(A - \frac{2}{n+1}\,\frac{Acc^T A^T}{c^T A c}\right)$$

When to stop?

Recall:

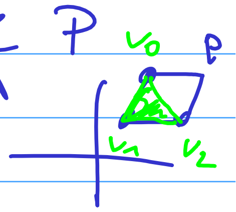
Fact 1: For an integer matrix $\overset{n\times n}{C}$: $|\det C| \le 2^{\langle C\rangle - n^2}$.

Fact 2: Let $T(x) = Q\cdot x + t$.
Then for $X \subseteq \mathbb{R}^n$, $\mathrm{vol}(T(X)) = |\det Q|\cdot \mathrm{vol}(X)$.

Lemma: If $\boxed{P = \{x \mid Cx \le d\}}$ has full dimension and C, d are integral, then

$$\mathrm{vol}(P) \ge 2^{-(n+1)\langle C\rangle + n^3}$$

Proof (sketch): P full dimension $\Rightarrow$
exist $v_0, v_1, \dots, v_n$ affine independent vertices of P



$\mathrm{vol}(\mathrm{conv}(v_0, v_1, \dots, v_n)) \le \mathrm{vol}(P)$

Let $e_i = (0, \dots, \underset{\uparrow\, i\text{-th coordinate}}{1}, \dots, 0)$, $i = 1, \dots, n$

$$\mathrm{conv}(v_0, v_1, \dots, v_n) = T(\mathrm{conv}(0, e_1, \dots, e_n)),$$

where

$$T(x) = v_0 + \underbrace{\begin{pmatrix} | & | & & | \\ v_1 - v_0 & v_2 - v_0 & \cdots & v_n - v_0 \\ | & | & & | \end{pmatrix}}_{=H} \begin{pmatrix} \vdots \\ x \\ \vdots \end{pmatrix}$$

$T(e_i) = v_0 + v_i - v_0 = v_i$

$T(0) = v_0$

Fact 3: $\mathrm{vol}(\mathrm{conv}(0, e_1, \dots, e_n)) = \frac{1}{n!}$

Thus,

$$\mathrm{vol}(P) \ge \mathrm{vol}(\mathrm{conv}(v_0, v_1, \dots, v_n)) \underset{\text{Fact 2,3}}{=} \frac{1}{n!}|\det H| =$$

$$= \frac{1}{n!}\left|\det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ v_0 & & H & \end{pmatrix}\right| = \frac{1}{n!}\left|\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ v_0 & v_1 & \cdots & v_n \end{pmatrix}\right| =$$

add the first column to every other

by theory of polyhedra we know:

Consider a vertex v_i : v_i is a unique solution of a system $\boxed{C_i x = d_i}$ for a subsystem $C_i x \le d_i$ of $\boxed{Cx \le d}$

by Cramer's rule: $v_{i,j} = \frac{\det C_{ij}}{\det C_i}$ → C_i with j-th column replaced by d_i

regular integral matrix

$$= \frac{1}{n!} \cdot \frac{\left|\det\begin{pmatrix} \det C_0 & \det C_1 & \cdots & \det C_n \\ u_0 & u_1 & \cdots & u_n\end{pmatrix}\right|}{|\det C_0 \cdot \det C_1 \cdots \det C_n|} \qquad \ge 1$$

where $u_i := v_i \cdot \det C_i$

Fact 1

$$\ge \frac{1}{n!}\left(\frac{1}{2^{\langle C\rangle - n^2}}\right)^{n+1} \ge 2^{-n\log n}\cdot 2^{-(n+1)\langle C\rangle + n^3}$$

☒

Theorem: If $P = \{x \mid Cx \le d\}$ is full dimensional and non-empty, for C, d integral, then the algorithm will find $x \in P$ in at most

→ $k = 2(n+1)(2(n+1)\langle c\rangle + n\langle d\rangle - n^3)$

Proof-hints: $\mathrm{vol}(E_0) \le (2R)^n$ for $R = \sqrt{n}\cdot 2^{\langle C,d\rangle - n^2}$

$\mathrm{vol}(P) \ge \ldots$ by previous lemma

$\mathrm{vol}(E_{k+1}) \le e^{\frac{-1}{2(n+1)}} \cdot \mathrm{vol}(E_k)$

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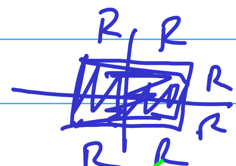
How to get rid of the simplifying assumptions

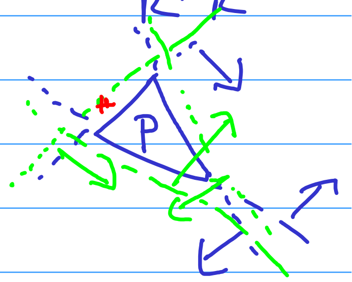
- P is bounded: replace P by $P \cap \{x \mid -R \le x \le R\}$

⋮



- P is of full dimension

Lemma 4.3: $Ax \le b$ has a solution iff $Ax \le b + \mathbb{1}\cdot\varepsilon$ has a solution for $\varepsilon = \frac{1}{2m \cdot 2^{\langle A|b\rangle - n^2}}$, where A, b are integral, $A \ldots m \times n$



Theorem: The LP problem is solvable in polynomial time.

Khachiyan, 1979

Final remarks:
- we ignore rounding issues –

$$f = a - \frac{1}{n+1} \cdot \frac{Ac}{\sqrt{c^T A c}} \quad \ldots \text{ irrational number}$$

- by the application of the Lemma ⚡, we only get YES/NO answer to the Decision version of LP, but there is a way how to get back to the search LP.

- the algorithm is not very efficient for real-life problems

- theoretically important as it can deal with enormously large system given a separation oracle – an algorithm that can check in poly time whether $x \in P$, and if not, find a violated constraint.