MATHEMATICAL PROGRAMMING

HISTORICAL SKETCH 23/11/2020

40's of 20th century : John von Neumann

1947 G. Dantzig ... simplex algorithm

60's interior point methods for general optimization problems
convergence ?

70's computational complexity – is simplex poly time ?

1972 Klee, Minty : No - exponential lower bound for simplex alg.

1979 Khachiyan - ellipsoid algorithm

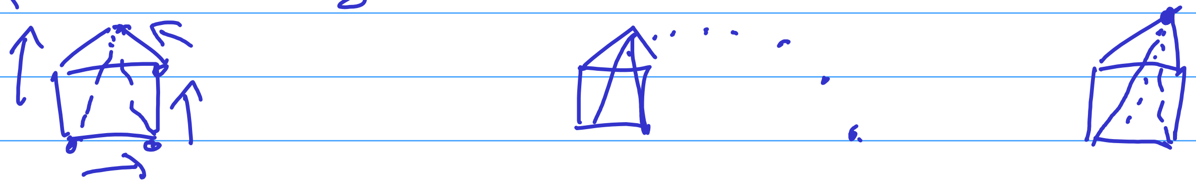
1984 Karmarkar - interior point method for LP

ELLIPSOID ALGORITHM

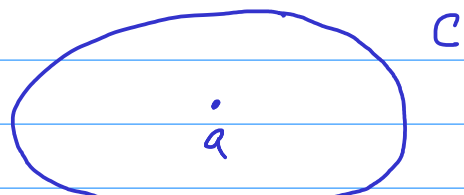
simplex ellipsoid interior p.m.

on the boundary outside inside

OPT

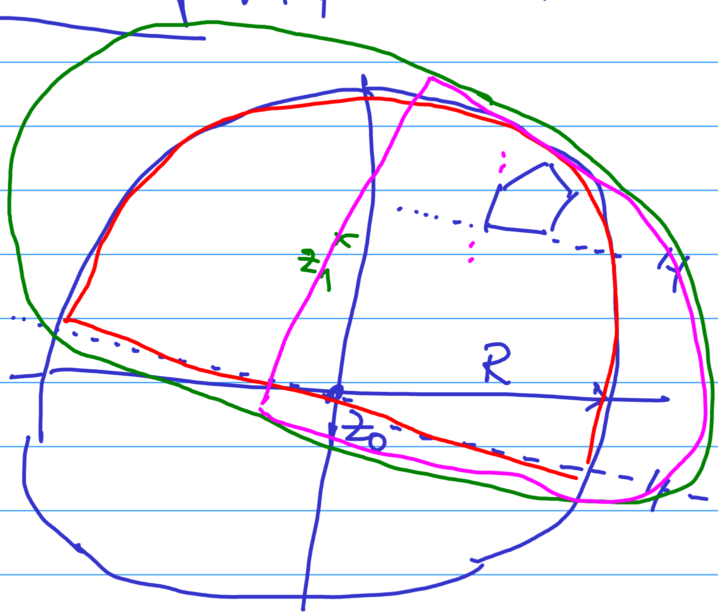


sketch of the algorithm



Two assumptions

- $P = \{x \mid Ax \le b\}$ is bounded
- if $P \neq \emptyset$ then P has full dimension $\mathbb{R}^n$



$z_k \overset{?}{\in} P$ YES → STOP

NO → smaller ellipsoid

Questions:

1. How to START – the first ellipsoid
2. Iteration – how to find the smaller ellipsoid
3. When to STOP, if $P \neq \emptyset$

three versions of the problem — poly time equivalent

1. Decision LP : $\exists x$ s.t. $Ax \le b$? YES / NO

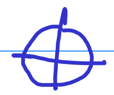
2. Search LP : find x s.t. $Ax \le b$, or declare "No such x"

3. Optimization LP : find x s.t. $Ax \le b$ that maximizes $c^T x$

Definitions

A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive definite if $\forall x \in \mathbb{R}^n, x \neq 0 : x^T A x > 0$

An ellipsoid with center in a given by $A \in \mathbb{R}^n$ is
$$E(a,A) = \{x \in \mathbb{R}^n \mid (x-a)^T A^{-1} (x-a) \le 1\}$$
E.g. $A = I$, $a = (0, \dots, 0)$



Claim : For any positive definite matrix A there exists a unique positive definite matrix Q s.t. $Q^T \cdot Q = A$

Notation : $A^{1/2} = Q$ ("square root of A")

Def : An affine transformation : $T : \mathbb{R}^n \to \mathbb{R}^n$ s.t. $T(x) = Q\cdot x + s$, where $Q \in \mathbb{R}^{n\times n}$, $s \in \mathbb{R}^n$

Note :
$$E(a,A) = \{x \in \mathbb{R}^n \mid \underbrace{(x-a)^T (A^{-1/2})^T}_{y^T} \underbrace{A^{-1/2}(x-a)}_{=y} \le 1\} =$$
$x = A^{1/2} y + a$
$$= \{A^{1/2} y + a \mid y^T \cdot y \le 1\} = A^{1/2} E(0, I) + a$$

$\Rightarrow$

Claim : Every $\overbrace{\text{ellipsoid}}^{E(a,A)}$ is an image of a unite ball centered at the origin, for $T(x) = A^{1/2}x + a$.

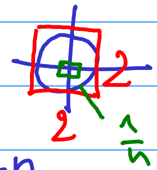
Fact: For an affine transformation $T(x) = Q \cdot x + t$ and a set $X \subseteq \mathbb{R}^n$:

$$vol(T(X)) = |\det Q| \cdot vol(X).$$

E.g. for ellipsoids

$$vol(E(a,A)) = |\det A^{1/2}| \cdot vol(E(0,I))$$

Observation: For an ellipsoid $E(a,A)$:



$$|\det A^{1/2}| \cdot n^{-n} \le vol(E(a,A)) \le |\det A^{1/2}| \cdot 2^n$$

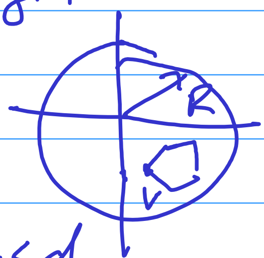
Input size: for an integer k: $\langle k \rangle = 1 + \lceil \log_2(|k|+1) \rceil$

$r = \frac{p}{q}$ .. $\langle r \rangle = \langle p \rangle + \langle q \rangle$

$a = (a_1, \dots, a_n)$ .. $\langle a \rangle = \sum_{i=1}^{n} \langle a_i \rangle$; $\langle C \rangle = \sum_{i,j} \langle c_{ij} \rangle$

Fact: For an integer matrix C ($n \times n$): $|\det C| \le 2^{\langle C \rangle - n^2}$

Lemma: If $P = \{x \mid Cx \le d\} \subseteq \mathbb{Q}^n$ is bounded and C and d are integral, then all vertices of P are contained in a ball centered at origin and radius $R = \sqrt{n} \cdot 2^{\langle C,d \rangle - n^2}$.



Proof: consider a vertex v of P

we know: 1) exists a subsystem $C'x \le d'$ of $Cx \le d$ s.t. v is a unique solution of $C'x = d'$

2) $\forall i$: $v_i = \frac{\det(C'_i)}{\det(C')}$ where C'_i is C' with i-th column replaced by d' — by Cramer's rule

3) $|\det(C')| \ge 1$ — C' is a regular, integral

$|\det(C'_i)| \le 2^{\langle C'_i \rangle - n^2} \Longrightarrow |v_i| \le 2^{\langle C'_i \rangle - n^2} \le 2^{\langle C,d \rangle - n^2}$

$\Rightarrow \|v\| \le \sqrt{n} \cdot 2^{\langle C,d \rangle - n^2} = R$ ∎

$\Rightarrow$ As the first ellipsoid, under our assumptions, use $E(0, R \cdot I)$.