EIGENVALUES AND EIGENVECTORS

LA 2 | 14 | 4 | 2016

Recall: For a polynomial $p(x)$ the multiplicity of the root r is the largest m s.t. $(x-r)^m$ divides $p(x)$ (i.e., $p(x) = (x-r)^m \cdot Q(x)$ for some $Q(x)$, $Q(r) \neq 0$)

Def: The algebraic multiplicity of an eigenvalue λ of $A \in \mathbb{C}^{n \times n}$ is the multiplicity of the root λ of $p_A(x)$.

The geometric multiplicity of an eigenvalue λ of $A \in \mathbb{C}^{n \times n}$ is $\dim(\text{Null}(A - \lambda I))$, i.e., the number of LI eigenvectors to λ.

LI .. linearly independent

Lemma: for each eigenvalue λ of $A \in \mathbb{C}^{n \times n}$, the geometric multipl. of λ $\leq$ the algebraic multipl. of λ.

Proof (sketch): Let $v_1, \dots, v_k$ be LI eigenvectors to λ. Extend them to a basis $v_1, \dots, v_n$ of $\mathbb{C}^n$ and let $R = (v_1 \dots v_n)$.

Then $\exists$ $D = \begin{pmatrix} \lambda I & B \\ 0 & C \end{pmatrix}$ s.t. $A \cdot R = R \cdot D \Rightarrow p_A(t) = p_D(t) = (t-\lambda)^k \cdot p_C(t)$
i.e., the algebraic mult. of λ is $\geq k$. ∎

Theorem: Matrix $A \in T^{n \times n}$ is diagonalizable $\Longleftrightarrow$
i) the sum of algebraic multiplicities of eigenvalues is n,
and ii) the geometric multiplicity of each eigenvalue equals its algebraic multiplicity. ($\Leftrightarrow$ the sum of geom. multipli. is n).

Note: For $T = \mathbb{C}$ we get: A is diagonalizable $\Longleftrightarrow$ condition ii) holds.
as i) holds always in $\mathbb{C}$

Proof: $\Rightarrow$ we know (by def.): $\exists$ invertible R and diagonal D s.t. $D = R^{-1} \cdot A \cdot R$, i.e., $AR = RD$, $\boxed{A}\boxed{R} = \boxed{R}\boxed{\ddots}$

i.e., $\forall i$, the i-th column of R is an eigenvector to D_{ii}

- the number of appearances of each eigenvalue equals its algebraic multiplicity
- For each appearance of an eigenvalue, there is a different eigenvector
- because R is invertible, the eigenvectors corresponding to the same eigenvalue are lin. independent
$\Rightarrow$ the geometric mult. $\geq$ the algebraic mult.

and by Lemma, they are equal.

⇐ Assume $\lambda_1, \dots, \lambda_k$ are eigenvalues of A of algebraic multiplicity $r_1, \dots, r_k$, and for each i, $b_i^1, \dots, b_i^{r_i}$ are <u>linearly independent</u> vectors corresponding to λ_i.

<u>Goal</u>: to show: $b_1^1, \dots, b_1^{r_1}, \dots, b_k^1, \dots, b_k^{r_k}$ are lin. independent.

Then they form a basis as $\sum_{i=1}^{k} r_i = n$, and by THM B, previous lecture the matrix A is diagonalizable.

$$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} b_1^1 \cdots b_k^{r_k} \end{bmatrix} = \begin{bmatrix} b_1^1 \cdots b_k^{r_k} \end{bmatrix} \begin{bmatrix} \overbrace{\lambda_1 \ \ }^{r_1} & & \\ & \ddots & \\ & & \overbrace{\ \ \lambda_k}^{r_k} \end{bmatrix}$$

<u>By contradiction</u>: if they are lin. depen., then exist α_i^j, <u>not all zero</u>, s.t.

$$\sum_{i=1}^{k} \underbrace{\left(\alpha_i^1 \cdot b_i^1 + \dots + \alpha_i^{r_i} \cdot b_i^{r_i} \right)}_{\text{denote } w_i} = 0 \qquad (*)$$

<u>Observe</u>: if $w_i = 0$ then $\alpha_i^1 = \alpha_i^2 = \dots = \alpha_i^{r_i} = 0$ as $b_i^1, \dots, b_i^{r_i}$ are LI by assumption of the THM

$\Rightarrow$ there must <u>exist</u> i s.t. <u>$w_i \neq 0$.</u>

Consider non-zero w_i: <u>w_i is an eigenvector to λ_i</u>

(last lecture - closed under $+, \cdot$)

Let $w_{j_1}, \dots, w_{j_\ell}$ be all <u>non-zero</u> w_i's. (last lecture)

As each of them corresponds to a different eigenvalue, they are LI —— a contradiction ↯

On the other hand, $\sum_{i=1}^{\ell} w_{j_i} = 0$ (by $(*)$) - they are lin. dep. ∎

DIAGONALIZATION OF SYMMETRIC MATRICES

We know: every $A\in\mathbb{C}^{n\times n}$ is similar to an upper triang. matrix

We show: every symmetric $A\in\mathbb{R}^{n\times n}$ is diagonalizable.

key notions: for $z=a+bi\in\mathbb{C}$, the complex conjugate is

$$\bar{z}=a-bi$$

$\forall z\in\mathbb{C}:\ z\cdot\bar{z}\in\mathbb{R}$

$\forall p,q\in\mathbb{C}:\ \overline{p\cdot q}=\bar{p}\cdot\bar{q}$ HW.

$$\overline{(a+bi)(c+di)}=\overline{ac-bd+i(ad+bc)}=ac-bd-i(ad+bc)=(a-bi)(c-di)$$

Def: The Hermitian (or: conjugate) transpose of $A\in\mathbb{C}^{m\times n}$ is the matrix $A^H\in\mathbb{C}^{n\times m}$, where $(A^H)_{ij}=\overline{A_{ji}}$.

A generalization of transpose: for $A\in\mathbb{R}^{m\times n}$, $A^H=A^T$.

E.g. For $A=\begin{pmatrix}2 & 1+i\\ -2i & 3+i\end{pmatrix}$, $A^H=\begin{pmatrix}2 & 2i\\ 1-i & 3-i\end{pmatrix}$.

Def: A matrix $A\in\mathbb{C}^{n\times n}$ is Hermitian, if $A^H=A$.

E.g. For $B=\begin{pmatrix}1 & 1+i\\ 1-i & 2\end{pmatrix}$, $B^H=B$, i.e., B is Hermitian.

1: If $A\in\mathbb{C}^{n\times n}$ is Hermitian, then $\forall i$, $A_{ii}\in\mathbb{R}$.

Proof: $\forall i$ we have: $A_{ii}=A^H_{ii}=\overline{A_{ii}}\Rightarrow A_{ii}\in\mathbb{R}$. ■

Lemma: $(AB)^H=B^H\cdot A^H$. Proof: similarly as for transposition: HW $((AB)^H)_{ij}\overset{?}{=}(B^H A^H)_{ij}$

(*) **Theorem**: If $A\in\mathbb{C}^{n\times n}$ is Hermitian, then each eigenvalue is real.

Proof: observe: $\forall v\in\mathbb{C}^n:\ v^Hv\in\mathbb{R}$ by above

for an eigenvalue λ of A and its eigenvector v:

$$\underbrace{(v^HAv)^H}_{z^H}\underset{\text{Lemma above}}{=}v^HA^Hv\underset{A^H=A}{=}\underbrace{v^HAv}_{z}=\underbrace{v^H\lambda v}_{\in\mathbb{R}}=\lambda\underbrace{\overbrace{v^Hv}^{\in\mathbb{R}}}_{\neq 0}\Rightarrow\lambda\in\mathbb{R}\ ■$$

$z^H = z \Rightarrow \in\mathbb{R}$

Theorem (Diagonalization of symmetric matrices, spectral decomposition)

Every symmetric $A \in \mathbb{R}^{n\times n}$ is diagonalizable.

Remark: holds more generally for Hermitian matrices.

We know: for each $v \in \mathbb{R}^n$ s.t. $v^T v = 1$ $\exists\, v_1 = v, v_2, \dots, v_n$ orthonormal basis

satisfying $\forall i,\ v_i^T v_i = 1$, $\forall i \neq j,\ v_i^T \cdot v_j = 0$

For $R = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$ it holds $R^T \cdot R = I$, i.e. $R^T = R^{-1}$. orthogonal matrix

Note: The proof is similar to the proof of similarity with triangul. matrices.

Proof: By induction on n. Base case $n = 1$ - trivial. ✓

Inductive step: $n-1 \longrightarrow n$

Consider symmetric $A \in \mathbb{R}^{n\times n}$ and an eigenvalue $\lambda \in \mathbb{R}$ (by THM ✱) and a corresponding real eigenvector $v \in \mathbb{R}^n$ (note: always possible)

Wlog assume $v^T v = 1$ (otherwise use $v' = \frac{v}{\sqrt{v^T v}}$)

Let $v_1 = v, v_2, \dots, v_n \in \mathbb{R}^n$ be an orthonormal basis of $\mathbb{R}^n$, and let $R = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$.

Then $\exists\, D \in \mathbb{R}^{n\times n}$ s.t. $A \cdot R = R \cdot D$, where $D = \left(\begin{array}{c|c} \lambda & z \\ \hline 0 & \\ \vdots & \bar{D} \\ 0 & \end{array}\right)$ (need not be diagonal)

i.e. $D = R^{-1} A R = R^T A R$

$A = R D R^{-1}$

Observe: $D^T = (R^T A R)^T = (R^T A^T R) = R^T A R = D$ - symmetric $\Rightarrow$

$\Rightarrow$ i) $z = 0$, and ii) $\bar{D}^T = \bar{D}$ is symmetric.

By inductive assump. $\exists\, \bar{C} \in \mathbb{R}^{(n-1)\times(n-1)}$ s.t. $\bar{T} = \bar{C}^{-1} \cdot \bar{D} \cdot \bar{C}$ is diagonal.

Then

$$D = \overbrace{\left(\begin{array}{c|c} \lambda & 0 \cdots 0 \\ \hline 0 & \\ \vdots & \bar{D} \\ 0 & \end{array}\right)}^{} = \overbrace{\left(\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & \bar{C} \\ 0 & \end{array}\right)}^{C} \overbrace{\left(\begin{array}{c|c} \lambda & 0 \cdots 0 \\ \hline 0 & \\ \vdots & \bar{T} \\ 0 & \end{array}\right)}^{T} \overbrace{\left(\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & \bar{C}^{-1} \\ 0 & \end{array}\right)}^{C^{-1}}$$

$$\Rightarrow A = R D R^{-1} = \underbrace{R C}_{=P} \overbrace{\,C\, T\, C^{-1}}^{=D} \underbrace{C^{-1} R^{-1}}_{=P^{-1}} = P \cdot \overset{\text{diagonal}}{T} \cdot P^{-1}$$

invertible, as both R, C are invertible ∎

orthogonal

Note: i) if $\bar{C}^T\bar{C} = I$, then

$$C^T\cdot C = \begin{bmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & \bar{C}^T \\ 0 & \end{bmatrix} \begin{bmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & \bar{C} \\ 0 & \end{bmatrix} = \begin{bmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & \bar{C}^T\bar{C} \\ 0 & \end{bmatrix} = I$$

for $A \in \mathbb{R}^{1\times 1}$, we can choose $A = \underset{\text{ortogonal}}{I}\cdot \overset{\text{diagon.}}{A} \cdot I$

ii) If R and C are both orthogonal, then $P = R\cdot C$ is orthogonal as well.

proof: $(R\cdot C)^T\cdot(RC) = C^T R^T R C = I$

Corollary: For every symmetric $A \in \mathbb{R}^{n\times n}$ there exist orthogonal P and diagonal T s.t. $A = P\cdot T\cdot P^{-1}$.

Note 2: The theorem is about real matrices, but the proof needs complex numbers (at the beginning - eigenvalue).