EIGENVALUES AND EIGENVECTORS

LA 2 | 7 | 4 | 2016

Def: Matrices A and A' – similar if $\exists R$ s.t. $A' = R \cdot A \cdot R^{-1}$

Def: $A \in T^{n \times n}$ – diagonalizable if similar to diagonal mat.

Def: $\lambda \in T$ – eigenvalue of $A \in T^{n \times n}$ if $\exists v \in T^n, v \neq 0$ s.t. $Av = \lambda v$

An eigenvector corresp. to λ is every $v \in V, v \neq 0$, s.t. $Av = \lambda v$.

Def: Characteristic polynomial of the matrix $A \in T^{n \times n}$ is the polynomial $p_A(t) = \det(A - t \cdot I)$ where $I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$.

THM A: For $A \in T^{n \times n}$, λ is an eigenvalue of $A \iff p_A(\lambda) = 0$.

Cor: A is singular $\iff 0$ is root of $p_A(t)$. ← new

THM B: $A \in T^{n \times n}$ diagonalizable $\iff \exists$ basis B of T^n of eigenvectors of A

THM C: If $\lambda_1, \dots, \lambda_n$ are pairwise distinct eigenvalues of A and $v_1, \dots, v_n$ eigenvectors corresponding to $\lambda_1, \dots, \lambda_n$, then $v_1, \dots, v_n$ are lin. independent.

Cor 1: An $n \times n$ matrix has at most n distinct eigenvalues.

Cor 2: If A has n dist. eigenvalues, then A is diagonalizable.

THM D: If A and B similar, then $p_A(t) = p_B(t)$ ⇒ the same eigenvalues

Def: Characteristic polynomial of the lin. trans. $f: V \to V$ is the char. polynomial of the matrix of f w.r.t. any basis.

For char. pol. $p_A(t) = b_n \cdot t^n + b_{n-1} \cdot t^{n-1} + \dots + b_1 t + b_0$ of A,

$$b_n = (-1)^n, \quad b_{n-1} = (-1)^{n-1} \cdot \sum_{i=1}^{n} a_{ii}, \quad b_0 = p_A(0) = \det(A).$$

Fact: Let p and q be polynomials. Then there exist unique polynomials s and r s.t. $p = q \cdot s + r$ and the degree of r (reminder) is smaller then the degree of q.

The fundamental THM of algebra (FTA): Every non-constant polynomial with complex coefficients has a complex root. (without proof)

Corollary (*): For every polynomial p of degree n there exist complex numbers $a \neq 0$, $\underbrace{a_1, \dots, a_n}_{\text{roots of } p}$ s.t.

$$p(x) = a(x - a_1)(x - a_2) \dots (x - a_n).$$

Note: in $\mathbb{R}$ not always possible! $p(x) = x^2 + 1$

Theorem E: Every matrix $A \in \mathbb{C}^{n\times n}$ is similar to an upper triangular matrix.

Proof: by induction on n: for $n=1$ it holds - $A \dots 1\times 1$

inductive step $n-1 \to n$: Consider $A \in \mathbb{C}^{n\times n}$

By FTA, there is $\lambda \in \mathbb{C}$ s.t. $p_A(\lambda) = 0$.

By THM A, λ is an eigenvalue of A.

Let v be the corresponding eigenvector: $Av = \lambda v$

Extend v to a basis B of $\mathbb{C}^n$: $\underbrace{v_1 = v, v_2, \dots, v_n}_{B}$, and

let $C = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_n \\ | & | & & | \end{pmatrix}$.

Consider the product $A\cdot C$ – its first column is λv.

As B (i.e., columns of C) is a basis, there exists $D \in \mathbb{R}^{n\times n}$ s.t. $A\cdot C = C \cdot D$ where $D = \left(\begin{array}{c|c} \lambda & z \\ \hline 0 & \bar{D} \end{array}\right)$,

"every column of AC is a lin. comb. of C-columns"

i.e., $A = C\cdot D \cdot C^{-1}$.

By ind. assumption, there exists $\bar{C} \in \mathbb{C}^{(n-1)\times(n-1)}$ s.t. $\underbrace{\bar{C}^{-1}\cdot \bar{D} \cdot \bar{C}}$ is an upper triangular matrix.

Let $\bar{T} =$, i.e. $\bar{D} = \bar{C}\cdot\bar{T}\cdot\bar{C}^{-1}$.

HW: Verify matrix multiplication

Thus

$$D = \left(\begin{array}{c|c} \lambda & z \\ \hline 0 & \bar{D} \end{array}\right) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \bar{C} \end{array}\right) \left(\begin{array}{c|c} \lambda & z\bar{C} \\ \hline 0 & \bar{T} \end{array}\right) \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \bar{C}^{-1} \end{array}\right)$$

$$\Rightarrow A = \underbrace{C \cdot \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \bar{C} \end{array}\right)}_{=R} \left(\begin{array}{c|c} \lambda & z\bar{C} \\ \hline 0 & \bar{T} \end{array}\right) \underbrace{\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \bar{C}^{-1} \end{array}\right) \cdot C^{-1}}_{=R^{-1}}$$

$\blacksquare$

Theorem: For a matrix $A \in \mathbb{C}^{n \times n}$ with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ it holds: i) $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$

ii) $\sum_{i=1}^{n} A_{ii} = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.

Proof: i) By the Corollary ⊗ of FTA and THM A,

$p_A(x) = (-1)^n (x - \lambda_1)(x - \lambda_2) \dots (x - \lambda_n)$

$$\det(A) \underset{\substack{\uparrow \\ \text{def. of } p_A(t)}}{=} p_A(0) = (-1)^n (-\lambda_1)(-\lambda_2) \dots (-\lambda_n) = \lambda_1 \cdot \lambda_2 \cdot \dots \cdot \lambda_n.$$

ii) Let T be an upper triangular matrix similar to A. ← by THM E

Then a) by THM D, $p_A(x) = p_T(x)$, and A, T have the same eigenvalues

b) T has the eigenvalues on the diagonal (triangular)

c) the coefficient of x^{n-1} is $(-1)^{n-1}$ · "sum of diagonal elements"

$$\Rightarrow \sum_{i=1}^{n} A_{ii} \overset{a+c}{=} \sum_{i=1}^{n} T_{ii} \overset{b}{=} \sum_{i=1}^{n} \lambda_i$$

Recall: For a polynomial $p(x)$ the multiplicity of the root r is the largest m s.t. $(x-r)^m$ divides $p(x)$ (i.e., $p(x) = (x-r)^m \cdot Q(x)$ for some $Q(x)$, $Q(r) \neq 0$)

Def: The algebraic multiplicity of an eigenvalue λ of $A \in \mathbb{C}^{n \times n}$ is the multiplicity of the root λ of $p_A(x)$.

The geometric multiplicity of an eigenvalue λ of $A \in \mathbb{C}^{n \times n}$ is $\dim(\text{Null}(A - \lambda I))$, i.e., the number of LI eigenvectors to λ.

LI .. linearly independent

Ex. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $p_A(\lambda) = \det(A - \lambda I) = (1-\lambda)^2$ ← multiplicity 2 of the root 1

$\Rightarrow$ the only eigenvalue of A is $\lambda = 1$ with algebraic mult. 2

eigenvectors of A? $(A - \lambda I)x = 0$ $\quad \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\dim(\text{Null}(A)) = 1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$\Rightarrow x_2 = 0$, eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\Rightarrow$ geometric multiplicity of eigenvalue 1 is 1.

Theorem: Matrix $A \in T^{n\times n}$ is diagonalizable $\Longleftrightarrow$

i) the sum of algebraic multiplicities of eigenvalues is n, and ii) the geometric multiplicity of each eigenvalue equals its algebraic multiplicity.

i) holds always in $\mathbb{C}$

<u>Note</u>: For $T = \mathbb{C}$ we get: A is diagonalizable $\Longleftrightarrow$ condition ii) holds.

<u>Proof</u>: $\Rightarrow$ we know (by def.): $\exists$ invertible R and diagonal D s.t. $D = R^{-1} \cdot A \cdot R$, i.e., $AR = RD$, $\boxed{A}\boxed{R} = \boxed{R}\boxed{\ddots}$

i.e., $\forall i$, the i-th column of R is an eigenvector to D_{ii}

- the number of appearances of each eigenvalue equals its algebraic multiplicity
- For each appearance of an eigenvalue, there is a different eigenvector
- because R is invertible, the eigenvectors corresponding to the same eigenvalue are lin. independent. ✓

$\Leftarrow$ Assume $\lambda_1, \dots, \lambda_k$ are eigenvalues of A of algebraic multiplicity $r_1, \dots, r_k$, and for each i, $b_i^1, \dots, b_i^{r_i}$ are <u>linearly independent</u> vectors corresponding to λ_i.

Goal: to show: $b_1^1, \dots, b_1^{r_1}, \dots, b_k^1, \dots, b_k^{r_k}$ are lin. independent.

Then they form a basis as $\sum_{i=1}^{k} r_i = n$, and by THM B, the matrix A is diagonalizable.

$$\boxed{A}\boxed{b_1^1 \cdots b_k^{r_k}} = \boxed{b_1^1 \cdots b_k^{r_k}}\begin{bmatrix} \overbrace{\lambda_1}^{r_1} & & \\ & \ddots & \\ & & \overbrace{\lambda_k}^{r_k} \end{bmatrix}$$

By contradiction: if they are lin. depen., then $\exists \alpha_i^j$, <u>not all zero</u>, s.t.

$$\sum_{i=1}^{k} \underbrace{\left(\alpha_i^1 \cdot b_i^1 + \dots + \alpha_i^{r_i} \cdot b_i^{r_i} \right)}_{\text{denote } w_i} = 0 \qquad (*)$$

<u>Observe</u>: if $w_i = 0$ then $\alpha_i^1 = \alpha_i^2 = \dots = \alpha_i^{r_i} = 0$ as $b_i^1, \dots, b_i^{r_i}$ are LI by assumption of the THM

$\Rightarrow$ there must exist i s.t. $w_i \neq 0$.

Consider non-zero w_i : w_i is an eigenvector to λ_i (last lecture)

Let $w_{j_1}, \dots, w_{j_\ell}$ be all non-zero w_j's.

As each of them corresponds to different eigenvalues, they are LI (last lecture) — a contradiction ↯

On the other hand, $\sum_{i=1}^{\ell} w_{j_i} = 0$ (by ⊛) - they are lin. dep. ∎