EIGENVALUES AND EIGENVECTORS

Review: our goal !: For a given lin. tran. $f: V \to V$

- to find a basis B s.t. ${}_B[f]_B$ is simple ($\sim$ diagonal)
- to find directions unaffected by f

Def: Square matrices A and A' are similar if $\exists$ an invertible R s.t. $A' = R \cdot A \cdot R^{-1}$

Def: $A \in T^{n \times n}$ is diagonalizable if similar to diagonal mat.

Def: For $A \in T^{n \times n}$, $\lambda \in T$ is an eigenvalue of A if $\exists v \in T^n, v \neq 0$ s.t. $Av = \lambda v$.

An eigenvector corresp. to λ is every $v \in V$, $v \neq 0$, s.t. $Av = \lambda v$.

Similarly for lin. trans. $f: V \to V$.

For an eigenvalue λ of LT f (or of a matrix A), the set $U = \{u : f(u) = \lambda \cdot u\}$ is a vector subspace.

set of all eigenvectors corresponding to λ and 0 !

Proof: We have to check that U is closed under $+, \cdot$.

For any $u, v \in U$, $t \in T$, it holds:

$f(t \cdot u) = t \cdot f(u) = t \cdot \lambda \cdot u = \lambda (t \cdot u)$, i.e., $tu \in U$

$f(u+v) = f(u) + f(v) = \lambda \cdot u + \lambda v = \lambda(u+v)$, i.e., $u+v \in U$ $\blacksquare$

Theorem A (Characterization of eigenvalues & determinants)

For $A \in T^{n \times n}$, λ is an eigenvalue of $A \iff \det(A - \lambda I) = 0$.

Proof: λ is eigenvalue $\iff \exists v \neq 0$ s.t. $Av = \lambda v$ $= \lambda I v$

$\iff \exists v \neq 0 \ (A - \lambda I)v = 0 \iff A - \lambda I$ is singular

$\iff \det(A - \lambda I) = 0$ $\blacksquare$

Def: Characteristic polynomial of the matrix $A \in T^{n \times n}$ is the polynomial $p_A(t) = \det(A - t \cdot I)$ where $I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$.

Roots of $p_A(t)$ are the eigenvalues of A. by THM

Eg. $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$, $\chi_A(t) = \det\begin{pmatrix} 1-t & 2 \\ 3 & 2-t \end{pmatrix} = (1-t)(2-t) - 2\cdot 3 = t^2 - 3t - 4 = (t-4)(t+1)$

Theorem: Let $f: V \to V$ be a linear transformation. There exists a basis B of V s.t. ${}_B[f]_B$ is diagonal $\iff$ there exists a basis of V consisting of eigenvectors of f.

Proof: $\Rightarrow$ Let $B = (b_1, \dots, b_n)$ be a basis s.t. ${}_B[f]_B$ diagonal, i.e.,

$$\left(\begin{bmatrix} | \\ f(b_1) \\ | \end{bmatrix}_B \cdots \begin{bmatrix} | \\ f(b_n) \\ | \end{bmatrix}_B \right) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$ for some $\lambda_1, \dots, \lambda_n$,

thus, $\forall i$, $f(b_i) = \lambda_i \cdot b_i$ - B consists of eigenvectors. $\square$

$\Leftarrow$ Let $B = (b_1, \dots, b_n)$ be a basis s.t. each b_i is an eigenvector, i.e., $\forall i$ $\exists \lambda_i$ s.t. $f(b_i) = \lambda_i \cdot b_i$.

thus,

$${}_B[f]_B = \left(\underbrace{[f(b_1)]_B}_{=[\lambda_1 b_1]_B} \cdots \underbrace{[f(b_n)]_B}_{=[\lambda_n b_n]_B} \right) = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \blacksquare$$

Theorem (for matrices): $A \in T^{n\times n}$ is diagonalizable $\iff$ there exists a basis B of T^n consisting of eigenvectors of A.

Proof: $\Rightarrow$ $\exists$ diagonal D and invertible R s.t.

HW ? $D = R^{-1} \cdot A \cdot R \implies R \cdot D = AR$

Let b_i denote the i-th column of R. Then $D_{ii} \cdot b_i = Ab_i$, i.e., each b_i is an eigenvector of A, and $\{b_1, \dots, b_n\}$ is a basis of T^n.

$\Leftarrow$ Let $R = \begin{pmatrix} | & & | \\ b_1 & \cdots & b_n \\ | & & | \end{pmatrix}$. Then $A \cdot R = R \cdot D$ where D is a diagonal m. and D_{ii} is the eigenvector to b_i.

As R is invertible, R^{-1} exists and $A = R \cdot D \cdot R^{-1}$ $\blacksquare$

Note: A is diagonalizable - the matrix ${}_B[f]_B$ of $f(x) = Ax$, is diagonal for some basis B.

Theorem: If $\lambda_1, \dots, \lambda_n$ are pairwise distinct eigenvalues of a lin. transformation f (matrix A) and $v_1, \dots, v_n$ eigenvectors corresponding to $\lambda_1, \dots, \lambda_n$, then $v_1, \dots, v_n$ are lin. independent.

Proof: by induction on n: $n = 1$ - OK

inductive step $n-1 \to n$: Consider lin. comb. $\sum_{i=1}^{n} \alpha_i v_i = 0$ ⊛

Then
$$0 = f(0) = f\left(\sum_{i=0}^{n} \alpha_i v_i\right) = \sum_{i=1}^{n} \alpha_i f(v_i) = \sum_{i=1}^{n} \alpha_i \lambda_i v_i =$$
$$= \sum_{i=1}^{n} \alpha_i \lambda_i v_i - \underbrace{\lambda_n \left(\sum_{i=1}^{n} \alpha_i v_i\right)}_{=0} = \sum_{i=1}^{n-1} \alpha_i \underbrace{(\lambda_i - \lambda_n)}_{\neq 0} v_i$$

by ind. assump., $v_1, \dots, v_{n-1}$ are LI $\Rightarrow \alpha_i = 0 \ \forall i = 1, \dots, n-1$

By ⊛, $\alpha_n = 0$ as well, i.e., $v_1, \dots, v_n$ are LI. ∎

Corollary 1: An $n \times n$ square matrix has at most n distinct eigenvalues.

Corollary 2: If an $n \times n$ matrix has n distinct eigenvalues, then it is diagonalizable. (the opposite implication does not hold ... I).
(by the two previous THMs)

Theorem: If A and B are similar, then $p_A(t) = p_B(t)$.

Proof: by definition (simil.) $\exists R$ s.t. $B = R \cdot A \cdot R^{-1}$

$$p_B(t) = \det(B - tI) = \det(R \cdot A \cdot R^{-1} - t \cdot R \cdot I \cdot R^{-1})$$
$$= \det(R \cdot (A - t \cdot I) R^{-1}) = \det(R) \det(A - tI) \cdot \det(R^{-1})$$
$$= \det(A - tI) = p_A(t). \quad ∎$$

Beware! The converse does not hold! E.g., for $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $p_A(t) = p_B(t)$, but they are not similar: if they were, then $\exists R$ s.t. $A = R \cdot B \cdot R^{-1} = I$ - contradiction

Def: Characteristic polynomial of the lin. trans. $f: V \to V$ is the char. polynomial of the matrix of f w.r.t. any basis.

Note: any two such matrices are similar, and by the previous THM they all have the same char. polyn.
$\Rightarrow$ definition makes sense!

IMPORTANT COEFFICIENTS OF CHAR. POLYN. OF A

Consider $p_A(t) = b_n \cdot t^n + b_{n-1} \cdot t^{n-1} + \ldots + b_1 t + b_0 = \det(A - tI)$

$$\begin{pmatrix} a_{11}-t & & & \\ & a_{22}-t & & \\ & & \ddots & \\ & & & a_{nn}-t \end{pmatrix}$$

- $b_n = (-1)^n$ … the t^n can be obtained only from the product $\prod_{i=1}^{n}(a_{ii}-t)$
- $b_{n-1} = (-1)^{n-1} \cdot \sum_{i=1}^{n} a_{ii}$ … similarly, the t^{n-1} can be obtaind only from the product $\prod_{i=1}^{n}(a_{ii}-t)$: pick $(-t)$ $(n-1)$-times and a_{ii} once; by summing these n products we get b_{n-1}
- $b_0 = \det(A)$ …. use $t=0$ in $A - tI$,
  $\quad$ " $p_A(0)$ $\quad$ in $p_A(t)$

Theorem: A matrix A is singular $\Leftrightarrow$ 0 is its eigenvalue.

Proof: we know: A is singul. $\Leftrightarrow \det(A) = 0 \Leftrightarrow$
$\Leftrightarrow \det(A - 0 \cdot I) = 0 \overset{\text{THM A}}{\Longleftrightarrow} 0$ is eigenvalue of A. $\blacksquare$

note: by coeff. b_0 of p_A we know whether A is invertible or not