Review: Determinant

Def: $\det(A) = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \cdot \prod_{i=1}^{n} a_{i\pi(i)}$.

EROs!

THM 1: $\det(A) = \det(A^T)$

THM 2: det is a linear function of each of its rows, columns

THM 3: swap of two lines changes the sign of the determinant.

THM 4: i) $A \in T^{n\times n}$ is invertible iff $\det(A) \neq 0$.

ii) $\det(A)$ can by computed by Gaussian elimination.

iii) For every $A, B \in T^{n\times n}$: $\det(B\cdot A) = \det(B)\cdot \det(A)$.

Theorem (Cramer's rule): Let $A \in T^{n\times n}$ be an invertible matrix and $b \in T^n$. Then the unique solution x of $Ax = b$ satisfies: $\forall i \in \{1, \dots, n\}$: $x_i = \dfrac{\det(A_{i\to b})}{\det(A)}$

where $A_{i\to b}$ is the matrix obtained from A by replacing its i-th column by b.

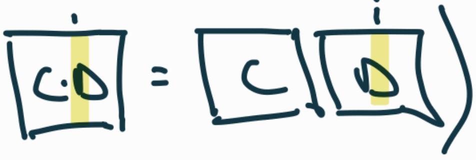
Proof: We know that the unique solution x satisfies $x = A^{-1}\cdot b$.

For $i \in \{1, \dots, n\}$, let $I_i := I_{i\to A^{-1}b} = \begin{array}{c} \begin{matrix} 1 & 2 & \cdots & i & \cdots & n \end{matrix} \\ \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \boxed{A^{-1}b} & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} \end{array}$,

i.e., I_i is obtained from I by replacing its i-th column by $A^{-1}b$.

Then
- $A_{i\to b} = A \cdot I_i$
- $\det(I_i) = x_i$.

(i-th column of $C\cdot D = C \cdot D_i$


)

$\Rightarrow$ By previous THM, part iii):

$\det(A_{i\to b}) = \det(A)\cdot \det(I_i) = \det(A)\cdot x_i$

i.e., $x_i = \dfrac{\det(A_{i\to b})}{\det(A)}$ ∎

A FORMULA FOR AN INVERSE MATRIX

Given A, find B s.t. $A \cdot B = I$.

- As $I = \begin{pmatrix} | & & | \\ e_1 & \cdots & e_n \\ | & & | \end{pmatrix}$, the solutions of $Ax = e_j$ for $j = 1, \dots, n$ (computation of j-th col of A^{-1}) give the columns of A^{-1}.
- For each e_j, by Cramer's rule:

$$x_i = \frac{\det(A_{i \to e_j})}{\det(A)}, \quad \text{where} \quad A_{i\to e_j} = \begin{array}{c} \text{i-th} \\ \begin{pmatrix} & 0 & & \\ & \vdots & & \\ \cdots & 1 & \cdots & \blacksquare \\ & \vdots & & \\ & 0 & & \end{pmatrix} \text{j-th} \\ e_j \qquad\quad k \end{array}$$

- $\det(A_{i\to e_j})$ does NOT depend on the (values of) row j of A:

Why: for each $\pi \in S_n$: if $\pi(j) \neq i$, then for some $\ell \neq j$, $\pi(\ell) = i$ $\Rightarrow$ the entry on row ℓ, column $\pi(\ell)$ is 0

i.e., only $\pi \in S_n$ s.t. $\pi(j) = i$ contribute to the det

$\Rightarrow$ Let A^{ji} denote the $(n-1)\times(n-1)$ submatrix of A (the minor of a_{ji}) obtained by the removal of row j and column i

Lemma: $\det(A_{i\to e_j}) = (-1)^{i+j} \cdot \det(A^{ji})$

Proof: Consider $(n-j)$ swaps of the row j of $A_{i\to e_j}$ with the row below it, and then $(n-i)$ swaps of the column i with the column to the right of it. We obtain the matrix $B = \begin{pmatrix} A^{ji} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \cdots & 1 \end{pmatrix}$.

Each of the swaps changes the sign of the det (THM 3).

$$\Rightarrow \det(A_{i\to e_j}) = (-1)^{n-i+n-j} \cdot \det(B) = (-1)^{i+j} \cdot \det(B).$$

$$\boxed{\det(B)} = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \cdot \prod_{k=1}^{n} B_{k\pi(k)} = \sum_{\substack{\pi \in S_n: \\ \pi(n) = n}} \operatorname{sgn}(\pi) \cdot \prod_{k=1}^{n-1} B_{k\pi(k)}$$

$$= \sum_{\pi \in S_{n-1}} \operatorname{sgn}(\pi) \cdot \prod_{k=1}^{n-1} B_{k\pi(k)} = \boxed{\det(A^{ji})}$$

$\blacksquare$

↓

Theorem: For an invertible $A \in T^{n\times n}$

$$(A^{-1})_{ij} = \frac{(-1)^{i+j} \det(A^{ji})}{\det(A)}.$$

Theorem (Laplace Expansion): For every $i \in \{1, \dots, n\}$:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ji} \cdot \det(A^{ji}).$$

Expansion by column i.

Proof: The i-th column of A equals $\sum_{j=1}^{n} A_{ji} \cdot e_i$.

By linearity of determinant, we get

$$\det(A) = \sum_{j=1}^{n} A_{ji} \cdot \det(A_{i \to e_j}) \underset{\text{Lemma } \square}{=} \sum_{j=1}^{n} (-1)^{i+j} a_{ji} \det(A^{ji}) \quad \blacksquare$$

Def: The adjugate of $A \in T^{n \times n}$, $n \geq 2$, is the matrix

$$\left(\operatorname{adj}(A)\right)_{ij} = (-1)^{i+j} \cdot \det(A^{ji}) \qquad i = 1, \dots, n, \; j = 1, \dots, n.$$

Note: For an invertible $A \in T^{n \times n}$: $A^{-1} = \dfrac{\operatorname{adj}(A)}{\det(A)}$

EIGENVALUES AND EIGENVECTORS

- Consider linear transformation $f: \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x) = A \cdot x$ for $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} = {}_K[f]_K$
- Note: A is also the matrix of f w.r.t. canonical basis K.
- Consider vectors $\boxed{b_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$, $\boxed{b_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}}$, basis $B = (b_1, b_2)$ and the matrix of f w.r.t the basis B.

By definition:

$${}_B[f]_B = \left([f(b_1)]_B , [f(b_2)]_B \right) =$$

$$= \left(\left[\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right]_B , \left[\begin{pmatrix} 8 \\ 12 \end{pmatrix}\right]_B \right) = \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}}_{=A'}$$

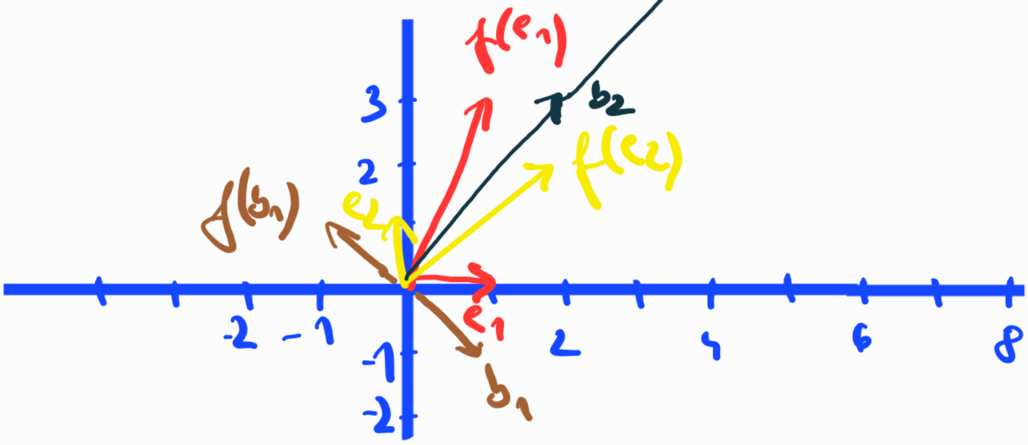
simple
"
a nice matrix -
- diagonal

connections

Observe:

$$\left.\begin{aligned} f(b_1) &= -b_1 \\ f(b_2) &= 4 \cdot b_2 \end{aligned}\right\}$$ i.e., b_1, b_2 is mapped onto its multiple

$f(b_2) = \begin{pmatrix} 8 \\ 12 \end{pmatrix}$



Note: $\det(A) = \det(A') = -4$

What's our goal? For a given lin. tran. $f: V \to V$ (or for a square matrix A)

- to find a basis B s.t. ${}_B[f]_B$ is simple ($\sim$ diagonal)
- to find directions unaffected by f

Recall: the transition matrix from $B' = (b_1', \dots, b_n')$ to B is
$$ {}_B[id]_{B'} = \begin{pmatrix} | & & | \\ [b_1']_B & \cdots & [b_n']_B \\ | & & | \end{pmatrix}. $$

It holds: ${}_{B'}[id]_B = {}_B[id]_{B'}^{-1}$, $\quad {}_B[f]_B = \underbrace{{}_B[id]_{KK}}_{T^{-1}}\,\underbrace{{}_{KK}[f]_{KK}}_{A}\,\underbrace{{}_{KK}[id]_B}_{T}$

In our case:
$$ \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}}_{A'} = \underbrace{\frac{1}{5}\begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}}_{T^{-1}} \underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}}_{T} $$
$\uparrow$ $\det T = 5$

Def: Square matrices A and A' are similar if there exists an invertible R s.t. $A' = R \cdot A \cdot R^{-1}$

Def: A matrix A is diagonalizable if it is similar to a diagonal matrix.

E.g. A and A' are similar, A is diagonalizable

Def: If $f: V \to V$ is a lin. trans. (V is VS over T), then $\lambda \in T$ is an eigenvalue of f if there exists non-zero vector $v \in V$ s.t. $f(v) = \lambda \cdot v$.

An eigenvector corresponding to λ is every $v \in V$, $v \neq 0$, s.t. $f(v) = \lambda \cdot v$.

Def: If $A \in T^{n \times n}$, then $\lambda \in T$ is an eigenvalue of A if there exists a non-zero vector v s.t. $Av = \lambda v$.
An eigenvector corresponding to λ is every $v \in V$, $v \neq 0$, s.t. $Av = \lambda v$.

For an eigenvalue λ of LT f (or of a matrix A), the set $U = \{u : f(u) = \lambda \cdot u\}$ is a vector subspace.
set of all eigenvectors corresponding to λ and 0 !

Proof: We have to check that U is closed under $+, \cdot$.
For any $u, v \in U$, $t \in T$, it holds:

$$f(t \cdot u) = t \cdot f(u) = t \cdot \lambda \cdot u = \lambda \cdot (t \cdot u)$$

$$f(u+v) = f(u) + f(v) = \lambda \cdot u + \lambda v = \lambda(u+v) \quad \blacksquare$$

Theorem (Characterization of eigenvalues & determinants)
For $A \in T^{n \times n}$, λ is an eigenvalue of $A \Leftrightarrow \det(A - \lambda I) = 0$.

Proof: λ is eigenvalue $\Leftrightarrow \exists v \neq 0$ s.t. $Av = \lambda v \quad = \lambda I v$

$$\Leftrightarrow \exists v \neq 0 \; (A - \lambda I)v = 0 \Leftrightarrow A - \lambda I \text{ is singular}$$

$$\Leftrightarrow \det(A - \lambda I) = 0 \quad \blacksquare$$

Def: Characteristic polynomial of the matrix $A \in T^{n \times n}$ is the polynomial $p_A(t) = \det(A - t \cdot I)$ where $I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$.

Roots of $p_A(t)$ are the eigenvalues of A.