Def: Determinant of a square matrix $A \in T^{n\times n}$

$$\det(A) = \sum_{\pi\in S_n} \mathrm{sgn}(\pi)\cdot \prod_{i=1}^{n} a_{i\pi(i)} .$$

Theorem:

i) If $A = \underset{\text{matrix}}{\begin{pmatrix} - & v_1 & - \\ & \vdots & \\ - & v_n & - \end{pmatrix}} \in T^{n\times n}$, $\underset{\text{vector}}{z^T \in T^n}$, $\underset{\text{scalars}}{\alpha, \beta \in T}$,

then for each $i \in \{1, \dots, n\}$

$$\det \overbrace{\begin{pmatrix} - & v_1 & - \\ & \vdots & \\ - & \alpha v_i + \beta z & - \\ & \vdots & \\ - & v_n & - \end{pmatrix}}^{=B} = \alpha\cdot\det(A) + \beta\cdot\det\begin{pmatrix} - & v_1 & - \\ & \vdots & \\ - & z & - \\ & \vdots & \\ - & v_n & - \end{pmatrix} \leftarrow i\text{-th row}$$

(i.e., det is a linear function of each of its rows)

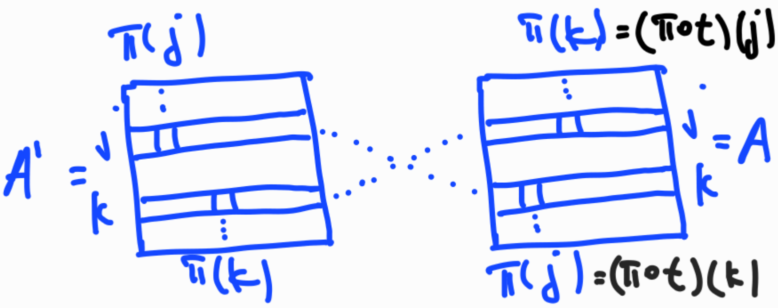
ii) If A' is obtained from A by a swap of two rows, then $\det(A') = -\det(A)$.

Proof:

ii) Let $j \neq k$ be the indices of the swapped rows, and $t \in S_n$ be a transposition $t(j)=k$, $t(k)=j$, $t(i)=i$ for $i \neq j,k$.

$$\det(A') \overset{\text{def.}}{=} \sum_{\pi\in S_n} \mathrm{sgn}(\pi)\cdot\prod_{i=1}^{n} a'_{i\pi(i)} = \sum_{\pi\in S_n} \mathrm{sgn}(\pi)\cdot\prod_{i=1}^{n} a_{i(\pi\circ t)(i)} =$$

$$= \sum_{(\pi\circ t)\in S_n} -\mathrm{sgn}(\pi\circ t)\prod_{i=1}^{n} a_{i(\pi\circ t)(i)} = \det(A)$$



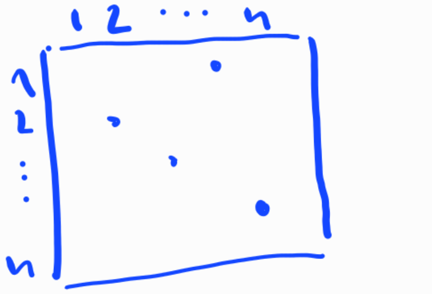
∎

Corollary: For a matrix A with two rows equal, $\det(A) = 0$.

For a triangular matrix A: $\det(A) = \prod_{i=1}^{n} a_{ii}$.

Theorem: $\det(A) = \det(A^T)$

Proof: Note that for each $\pi \in S_n$:



$$\prod_{i=1}^{n} a_{i\pi(i)} = \prod_{j=1}^{n} a_{\pi^{-1}(j)j}$$

$j = \pi(i) \Leftrightarrow \pi^{-1}(j) = i$

As $\operatorname{sgn}(\pi) = \operatorname{sgn}(\pi^{-1})$, we get:

$$\det(A) = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_{i=1}^{n} a_{i\pi(i)} = \sum_{\pi^{-1} \in S_n} \operatorname{sgn}(\pi^{-1}) \prod_{j=1}^{n} a_{\pi^{-1}(j)j}$$

$$= \sum_{\pi' \in S_n} \operatorname{sgn}(\pi') \sum_{j=1}^{n} A^T_{j\pi'(j)} = \det(A^T)$$

denote $\pi' := \pi^{-1}$

$\blacksquare$

Note: Part ii) (linearity) from the first theorem holds for columns, too.

DETERMINANTS AND EROs

Questions: 1. What are the determinants of matrices of EROs

2. how EROs change determinants

- multiplication of a row i by $t \neq 0$:

$$E = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & t & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \; i$$

1. $\det(E) = t$
2. by point i) from THM:

$$\det(E \cdot A) = t \cdot \det(A) = \det(E) \cdot \det(A)$$

- addition of $\overbrace{\text{of}}^{t\text{-multiplication}}$ row j to row $i \neq j$

$$\begin{array}{cc} & \begin{array}{ccccc} & & j & & \end{array} \\ E = & \begin{pmatrix} 1 & & & & \\ & 1 & & t & \\ & & \ddots & \vdots & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} \end{array}$$

1. $\det(E) = 1$
2. by point i) from THM & Corol:

$$\det(E \cdot A) = \det(A) = \det(E) \cdot \det(A)$$

- swap of two rows HW

$$\begin{array}{cc} & \begin{array}{cccccc} & & i & & j & \end{array} \\ E = & \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 0 & & 1 & \\ & & & 1 & & \\ & & 1 & & 0 & \\ & & & & & \ddots \; 1 \end{pmatrix} \begin{array}{c} \\ \\ i \\ \\ j \\ \\ \end{array} \end{array}$$

1. $\det(E) = -1$
2. by point ii) from THM:

$$\det(E \cdot A) = -\det(A) = \det(E) \cdot \det(A)$$

$\Rightarrow$ we get the following lemma:

Lemma: Let $A \in T^{n \times n}$ and E be a matrix of an ERO.

Then: $\det(E \cdot A) = \det(E) \cdot \det(A)$.

Consider a sequence of EROs and their matrices $E_1, \dots, E_k$, and a matrix A. Then, by the previous lemma:

$(*)$ $\det(E_k \cdot E_{k-1} \cdot \dots \cdot E_2 \cdot E_1 \cdot A) = \det(E_k) \cdot \det(E_{k-1} \cdots E_2 E_1 A) =$

$= \det(E_k) \cdot \det(E_{k-1}) \cdot \det(E_{k-2} \cdots E_2 E_1 A) = \dots =$

$= \det(E_k) \cdot \det(E_{k-1}) \cdots \det(E_2) \cdot \det(E_1) \cdot \det(A)$

Let $F = E_k E_{k-1} \cdots E_1$. By applying $(*)$ to $A = I$, we obtain

- $\det(F) = \det(E_k \cdots E_1) = \det(E_k) \cdots \det(E_1) \neq 0$!

in words: the matrix of any sequence of EROs is non-zero!

- $\det(F \cdot A) = \det(F) \cdot \det(A)$

Theorem: i) A matrix $A \in T^{n \times n}$ is invertible if and only if $\det(A) \neq 0$.

ii) $\det(A)$ can by computed by Gaussian elimination:

$$\det(A) = \frac{\det(F \cdot A)}{\det(F)}$$ ← triangular matrix

iii) For every $A, B \in T^{n \times n}$: $\det(B \cdot A) = \det(B) \cdot \det(A)$.

Proof: i) Let F be a matrix of EROs s.t. $F \cdot A$ is in RREF.

- if A is invertible, then $F \cdot A = I$.

$\Rightarrow 1 = \det(I) = \det(F \cdot A) \overset{\text{Lemma}}{=} \det(F) \cdot \det(A) \Rightarrow \det(A) \neq 0$

- if A is singular, there is a zero-row in $F \cdot A \Rightarrow$

$0 = \det(F \cdot A) \overset{\text{Lemma}}{=} \underbrace{\det(F)}_{\neq 0} \cdot \det(A) \Rightarrow \det(A) = 0$.

ii) by Lemma

iii) holds if B is a matrix of a sequence of EROs - Lemma

Note: Every invertible B is such a matrix

If B is singular, then $B \cdot A$ singular $\Rightarrow \underbrace{\det(B \cdot A)}_{=0} = \overbrace{\det(B)}^{=0} \cdot \det(A)$ ∎

Theorem (Cramer's rule): Let $A \in T^{n \times n}$ be an invertible matrix and $b \in T^n$. Then the unique solution x of $Ax = b$ satisfies: $\forall i \in \{1, \dots, n\}$:

$$x_i = \frac{\det(A_{i \to b})}{\det(A)}$$

where $A_{i \to b}$ is the matrix obtained from A by replacing its i-th column by b.

Proof: We know that the unique solution x satisfies $x = A^{-1} \cdot b$.

For $i \in \{1, \dots, n\}$, let

$$I_i := I_{i \to A^{-1}b} = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & \cdots & i & \cdots \; n \end{array} \\ \begin{pmatrix} 1 & & \boxed{A^{-1}b} & & 0 \\ & \ddots & & & \\ 0 & & & \ddots & 1 \end{pmatrix} \end{array},$$

i.e., I_i is obtained from I by replacing its i-th column by $A^{-1} \cdot b$.

Then
- $A_{i \to b} = A \cdot I_i$
- $\det(I_i) = x_i$.

(i-th column of $C \cdot D = C \cdot D_i$

$$\boxed{C \cdot D} = \boxed{C}\,\boxed{D}$$)

$\Rightarrow$ By previous THM, part iii):

$$\det(A_{i \to b}) = \det(A) \cdot \det(I_i) = \det(A) \cdot x_i$$

i.e., $$x_i = \frac{\det(A_{i \to b})}{\det(A)}$$