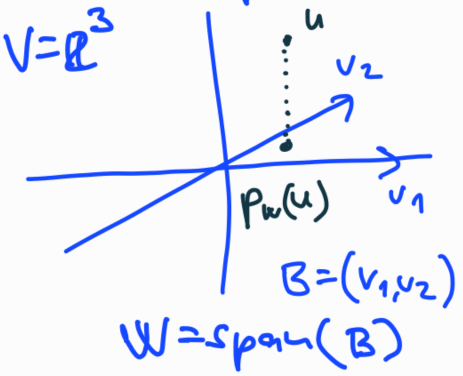
Review: LA 2 $\boxed{3/3/2026}$

Def: Let W be a subspace of VS V, $B=(v_1,\dots,v_n)$ an orthonormal basis of W.
The mapping $p_W(u)=\sum_{i=1}^{n}\langle u|v_i\rangle\cdot v_i$ from V to W is orthogonal projection of V onto the subspace W.



Lemma: $\forall j=1,\dots,n: (u-p_W(u))\perp v_j$

Lemma: $p_W(u)$ is the unique nearest vector from W to u.

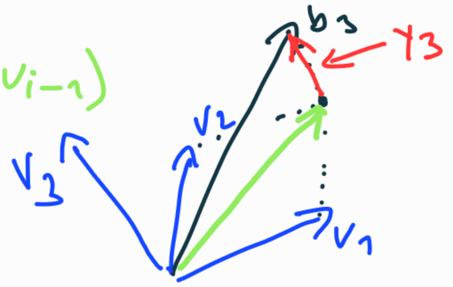
GRAM-SCHMIDT ORTHOGONALIZATION

given a basis $(b_1,\dots,b_n)$, it constructs an ON basis $(v_1,\dots,v_n)$.

for $i=1,\dots,n$ do

projection of b_i onto $\mathrm{span}(v_1,\dots,v_{i-1})$

$$(*)\quad \left[\begin{aligned} y_i &:= b_i-\sum_{j=1}^{i-1}\langle b_i|v_j\rangle\cdot v_j \\ v_i &= \frac{y_i}{\|y_i\|}\end{aligned}\right.$$



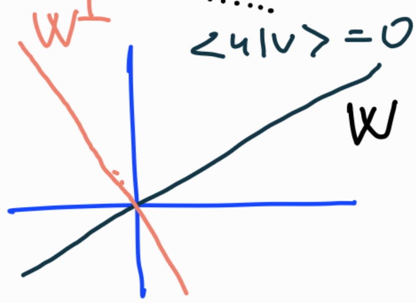
Corollary 1: Every finite dim VS has ON basis.

Corollary 2: In a finite dim. VS V: if W is a subspace of V, then every ON basis of W can be extended into an ON basis of V.

Proof: first extend the given ON basis, say $b_1,\dots,b_k$, of W, into a basis $b_1,\dots,b_k,z_1,\dots,z_\ell$ of V by Steinitz theorem.
Then Gram-Schmidt. Note: $\forall i=1,\dots,k: v_i=b_i$ .. no change! $\boxtimes$

Def: Let W be set of vectors from a VS V with inner product. Then the **orthogonal complement** of W is the set $W^\perp = \{ v \in V : \forall u \in W, u \perp v \}$.

$\langle u | v \rangle = 0$



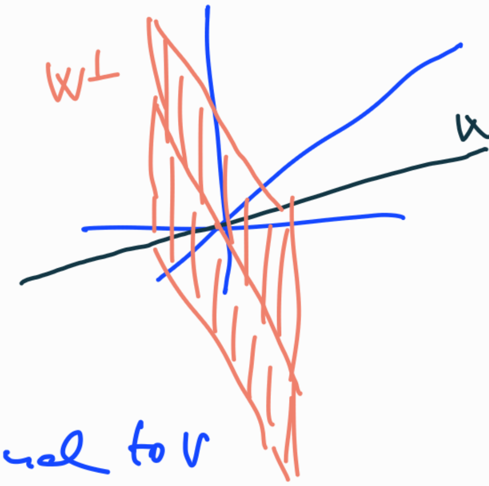
Eg. • $V = \mathbb{R}^2$, standard inner product

W ... line through the origin

$W^\perp$ ... perpendicular line through origin

• $V = \mathbb{R}^3$, W ... line through the origin

$W^\perp$ ... plane perpendicular to W, through origin



$U \subseteq V \Rightarrow V^\perp \subseteq U^\perp$

proof: obvious: "everything orthogonal to V is orthogonal to U as well".

$x \in V^\perp \iff \forall v \in V, x \perp v \Rightarrow \forall v \in U, x \perp v \iff x \in U^\perp$ ■

Consider a homogeneous system of lin. eq. $Ax = 0$. $A \in \mathbb{R}^{m \times n}$

Then $\mathrm{Null}(A) = (R(A))^\perp$. standard inner prod.

$\{ x \mid Ax = 0 \}$ row space of A

proof: $\supseteq$ ... by previous

$\subseteq$ ... consider $y \in \mathrm{Null}(A)$ and any $z \in R(A)$. i.e. $z = \sum_i \lambda_i A_{i*}$

We need to check that $\langle z | y \rangle = 0$:

$$\langle \overbrace{\sum_i \lambda_i A_{i*}}^{z} | y \rangle \overset{L1,2}{=} \sum_i \lambda_i \langle A_{i*} | y \rangle = 0 \Rightarrow y \in R(A)^\perp$$ ■

Lemma: If $B = (b_1, \dots, b_n)$ is a basis of subspace W, then $v \in W^\perp \iff \forall i = 1, \dots, n, v \perp b_i$.

Proof: $\Rightarrow$ obvious, by

$\Leftarrow$ we have to check that $\forall u \in W, \langle u | v \rangle = 0$

$u = \sum_{i=1}^n \alpha_i b_i$ for some α_i. $\langle u | v \rangle = \langle \sum_i \alpha_i b_i | v \rangle \overset{L1,2}{=} \sum_i \alpha_i \overbrace{\langle b_i | v \rangle}^{=0} = 0$. ■

Theorem (Properties of orthogonal complement).

Let W be a subspace of VS V of finite dimension.

Then i) $W^\perp$ is a subspace of V.

ii) $\dim(W) + \dim(W^\perp) = \dim(V)$

iii) $(W^\perp)^\perp = W$

iv) $W^\perp \cap W = \{0\}$.

Proof: i) we check that $W^\perp$ is closed under $+, \cdot$:

- $u, v \in W^\perp$: $\forall x \in W$, $\langle u+v | x\rangle = \underbrace{\langle u|x\rangle}_{=0} + \underbrace{\langle v|x\rangle}_{=0} = 0$, as $u, v \in W^\perp$

  $\Rightarrow u+v \in W^\perp$

- $u \in W^\perp$, $\alpha \in T$: $\forall x \in W$, $\langle \alpha u | x\rangle = \alpha \overbrace{\langle u|x\rangle}^{=0} = 0 \Rightarrow \alpha u \in W^\perp$

ii) Let $B = (b_1, \dots, b_k)$ be an ON basis of W. We extend it to an ON basis of V: $H = (b_1, \dots, b_k, c_1, \dots, c_\ell)$.

Now we verify that $C = (c_1, \dots, c_\ell)$ is a basis of $W^\perp$.

Consider any $v \in W^\perp$. Its coordinates w.r.t. H are given by the Fourier coefficients:

$$\left.\begin{array}{l} \overbrace{\langle v|b_i\rangle}^{=0} \dots \text{for the vectors } b_i \\ \langle v|c_j\rangle \dots \text{for the vectors } c_j \end{array}\right\} \Rightarrow v \in \mathrm{span}(C) \Rightarrow C \text{ is basis of } W^\perp.$$

iii) By Lemma, $v \in (W^\perp)^\perp \Leftrightarrow \forall i = 1, .., \ell$, $v \perp c_i$, $\Leftrightarrow$

$\forall i = 1, .., \ell$, $\langle v|c_i\rangle = 0 \Leftrightarrow v \in \mathrm{span}(B) = W$.

↑ Fourier coeff. w.r.t. the basis H of V

iv) If $\exists u \in W^\perp \cap W$, $u \neq 0$, then $\exists \alpha_1 \dots \alpha_k, \beta_1, \dots, \beta_\ell$ s.t. $\sum^k \alpha_i b_i = \sum^\ell \beta_j c_j \Rightarrow H$ is not a basis of V - contradiction. ∎

Or: by proof of ii): $\forall v \in V$ $\exists u \in W$, $z \in W^\perp$ s.t. $v = u + z$

$\Rightarrow W + W^\perp = V$

As $\dim(X+Y) + \dim(X \cap Y) = \dim(X) + \dim(Y)$,

$\dim(W \cap W^\perp) = (k+\ell) - k - \ell = 0$.

LEAST SQUARES METHOD

Consider an inconsistent (= no solution) system of linear equations $Ax = b$, $A \in \mathbb{R}^{m \times n}$.

Goal: find $x \in \mathbb{R}^n$ that violates the constraints $\overbrace{\text{the least}}^{\text{meaning?}}$.

Recall: $Ax = b$ - find a linear combination of colums of A that equals b

Let $W = C(A)$. No solution of $Ax = b$: $b \notin W$

Idea: 1) compute the projection $p_W(b)$ of b to W

2) and solve $Ax = p_W(b)$.

Then • there is a solution ... denote it $\bar{x}$

• moreover: $\underbrace{\|A\bar{x} - b\|}_{\text{measure of constraints violation}} = \|p_W(b) - b\| = \min_{y \in W} \|y - b\| = \min_{x \in \mathbb{R}^n} \overset{\text{standard norm}}{\|Ax - b\|}$

i.e., $\bar{x}$ is such that the norm of $A\bar{x} - \bar{b}$ is the least, as required.

Why least squares?

Note: the optima of $\min_{x \in \mathbb{R}^n} \|Ax - b\|$ and $\min_{x \in \mathbb{R}^n} \|Ax - b\|^2 =$

are the same (by monotonicity of z^2) $\quad = \underbrace{\min_{x \in \mathbb{R}} \sum_i (A_{i,*}x - b_i)^2}_{\text{least squares}}$

HOW TO SOLVE IT?

aim: avoid projection

Let $\mho = \{x \in \mathbb{R}^n \mid Ax = p_W(b)\}$.

Theorem: $\mho = \{x \in \mathbb{R}^n \mid A^T A x = A^T b\}$.

Proof: as $(b - p_W(b)) \perp W$, and $W = C(A) = R(A^T)$,

we have $b - p_W(b) \in \mathrm{Null}(A^T)$.

$\subseteq$ Consider $x \in \mho$: $Ax = p_W(b) \Rightarrow A^T(Ax - p_W(b)) = 0 \Leftrightarrow Ax - p_W(b) \in \mathrm{Null}(A^T) \Leftrightarrow$

$\Leftrightarrow \underbrace{Ax - p_W(b) + (p_W(b) - b)}_{= Ax - b} \in \mathrm{Null}(A^T) \Leftrightarrow A^T(Ax - b) = 0 \Leftrightarrow A^T A x = A^T b$.

$\supseteq$ The other inclusion: note that $\overset{\in C(A)}{Ax} - \overset{\in C(A)}{p_W(b)} \in C(A)$, and $Ax - p_W(b) \perp C(A)$ $\overset{\in C(A)^\perp}{}$

$\overset{\text{THM on }\perp}{\Rightarrow} Ax - p_W(b) = 0$, i.e., $x \in \mho$. ∎

$\Rightarrow$ A way to solve it.