INNER PRODUCT

24/2/2026

Review. **Def**: Inner product - a mapping $V \times V \to \mathbb{R}$ (or $\mathbb{C}$) satisfying the following axioms:

linear:

(L1) $\forall x, y \in V$, $\forall \alpha \in \mathbb{R}$ (or $\mathbb{C}$): $\langle \alpha x | y \rangle = \alpha \langle x, y \rangle$

(L2) $\forall x, y, z \in V$: $\langle x + y | z \rangle = \langle x | z \rangle + \langle y | z \rangle$

positive → (P) $\forall x \in V$: $\langle x | x \rangle \geq 0$ and $\langle x | x \rangle = 0$ only for $x = 0$

conjugat. → (C) $\forall x, y \in V$: $\langle x | y \rangle = \underbrace{\overline{\langle y, x \rangle}}_{\text{complex conjugate}}$ (i.e. $\langle x | y \rangle = \langle y, x \rangle$ for $\mathbb{R}$)

We observed:

- $\forall x, y, z \in V$: $\langle x | y + z \rangle = \langle x | y \rangle + \langle x | z \rangle$ (by (C), (L1), (C))
- $\forall x, y \in V$, $\forall \alpha \in \mathbb{C}$ $\langle x | \alpha y \rangle = \bar{\alpha} \langle x | y \rangle$ .... (L1')

Theorem (Cauchy-Schwarz inequality)

$\forall x, y \in V$: $|\langle x | y \rangle| \leq \underbrace{\sqrt{\langle x | x \rangle}}_{\|x\|} \underbrace{\sqrt{\langle y | y \rangle}}_{\|y\|}$ norm induced by $\langle \cdot | \cdot \rangle$

Def: Norm - a mapping $V \to \mathbb{R}$ (always to $\mathbb{R}$) satisfying the following axioms:

(P) $\forall x \in V$: $\|x\| \geq 0$, and $\|x\| = 0$ only for $x = 0$

(L) $\forall x \in V$, $\forall \alpha \in \mathbb{R}$ (or $\mathbb{C}$): $\|\alpha x\| = |\alpha| \cdot \|x\|$

(TI) $\forall x, y \in V$: $\|x + y\| \leq \|x\| + \|y\|$

ORTHOGONALITY

Def: Vectors x, y in a VS V with an inner product $\langle \cdot | \cdot \rangle$ are **orthogonal**, if $\langle x | y \rangle = 0$.

Notation: $x \perp y$

1 Every set of non-zero mutually orthogonal vectors is linearly independent.

Proof: given $v_1, \dots, v_k$ s.t. $\forall i \neq j$, $v_i \perp v_j$, assume, for a contradiction, that (wlog)

$$v_1 = \sum_{i=2}^{k} \alpha_i v_i$$

$$\text{Then } \langle v_1 | v_1 \rangle = \left\langle \sum_{i=2}^{k} \alpha_i v_i \,\Big|\, v_1 \right\rangle \overset{L1, L2}{=} \sum_{i=2}^{k} \alpha_i \underbrace{\langle v_i | v_1 \rangle}_{=0} = 0$$

$\Rightarrow$ by axiom (P), $v_1 = 0$ - a contradiction with $v_1 \neq 0$. ∎

Def: A **basis** $B = (v_1, \dots, v_n)$ of a VS V with an inner product is **orthonormal**, if

- $\forall i \neq j$, $v_i \perp v_j$ (ON basis)
- $\forall i$, $\|v_i\| = 1$.

Lemma: Let $(v_1, \dots, v_n)$ be an orthonormal basis of V, and $x \in V$. Then $x = \langle x | v_1 \rangle v_1 + \dots + \langle x | v_n \rangle v_n$.

Proof: Let $\alpha_1, \dots, \alpha_n$ be such that $x = \sum_{i=1}^{n} \alpha_i v_i$.

Then $\forall j$: $\langle x | v_j \rangle = \left\langle \sum_{i=1}^{n} \alpha_i v_i \,\Big|\, v_j \right\rangle = \sum_{i=1}^{n} \alpha_i \langle v_i | v_j \rangle = \alpha_j$ ∎

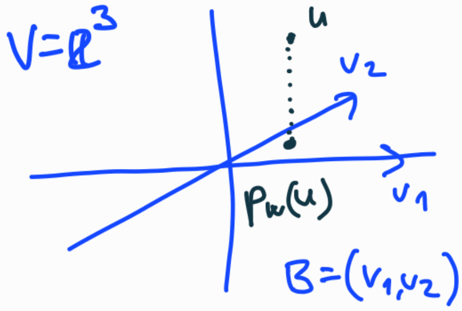
coefficients $\langle x | v_j \rangle$ .. Fourier coefficients

Meaning of Lemma: with orthonormal basis, it is easy to get coordinates (lin. combination)

Def: Let W be a subspace of VS V with inner product and $B=(v_1,\dots,v_n)$ an orthonormal basis of W. Then the mapping $p_W : V \to W$ defined by

$$p_W(u) = \sum_{i=1}^{n} \langle u | v_i \rangle \cdot v_i \quad \text{is orthogonal projection}$$

of V onto the subspace W.



HW: Verify that p_W is a linear transformation.

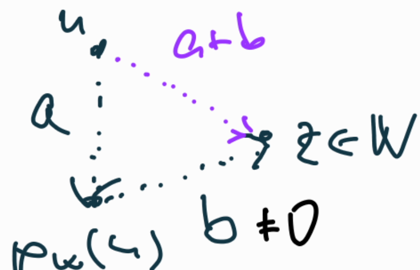
2 $\forall j = 1,\dots,n : (u - p_W(u)) \perp v_j$

Proof: $\langle u - p_W(u) | v_j \rangle = \langle u - \sum_{i=1}^{n} \langle u|v_i\rangle v_i | v_j \rangle \overset{L1,L2}{=}$

$$\langle u|v_j\rangle - \sum_{i=1}^{n} (\langle u|v_i\rangle \underbrace{\langle v_i|v_j\rangle}_{=1 \text{ if } i=j,\ =0 \text{ if } i\neq j}) = \langle u|v_j\rangle - \langle u|v_j\rangle = 0$$

Lemma: $\forall z \in W,\ z \neq p_W(u) : \|u - p_W(u)\| < \|z - u\|$ (i.e., $p_W(u)$ is the unique nearest vector from W to u).

Proof: Consider any $z \in W$, and the triangle $u, z, p_W(u)$.



Let $a = p_W(u) - u$, $b = z - p_W(u)$.

It suffices to check that $\|a\| \overset{?}{<} \|a+b\|$:

$$\|a+b\|^2 = \langle a+b | a+b \rangle = \langle a|a\rangle + \underbrace{\langle a|b\rangle}_{=0} + \underbrace{\langle b|a\rangle}_{=0} + \langle b|b\rangle \quad \text{by 2 } a \perp b$$

$$= \|a\|^2 + \overbrace{\|b\|^2}^{\neq 0} > \|a\|^2$$

Corollary: The projection mapping p_W is independent on the choice of the basis of W i.e., on the VS span(y)

The norm of a projection of x on a unit vector y is less or equal than $\|x\|$. By Cauchy Schwarz.

Proof: $\|\langle x|y\rangle \cdot y\| = |\langle x|y\rangle| \cdot \|y\| \le \|x\| \cdot \|y\| = \|x\|$

Key Question: Do orthonormal bases exist?

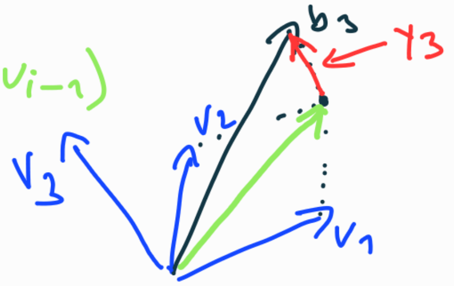
For VS of dimension 1 obviously yes - just scale.

GRAM-SCHMIDT ORTHOGONALIZATION: an algorithm that from a given basis $(b_1,\dots,b_n)$ constructs an orthogonal basis $(v_1,\dots,v_n)$.

for $i=1,\dots,n$ do

(*) $\quad y_i := b_i - \underbrace{\sum_{j=1}^{i-1} \langle b_i | v_j \rangle \cdot v_j}_{\text{projection of } b_i \text{ onto span}(v_1,\dots,v_{i-1})}$

$\quad v_i = \dfrac{y_i}{\|y_i\|}$



Theorem: $\overbrace{(v_1,\dots,v_n)}^{\text{constructed by the algorithm}}$ is an orthonormal basis of $\text{span}(b_1,\dots,b_n)$.

Proof: by induction we show:

$\forall i$: $(v_1,\dots,v_i)$ is an orthonormal basis of $\text{span}(b_1,\dots,b_i)$

base case $i=1$: $y_1 = b_1$, $v_1 = \dfrac{y_1}{\|y_1\|}$ $\Rightarrow \|v_1\| = 1$

as v_1 is a non-zero multiple of b_1, $\text{span}(b_1) = \text{span}(v_1)$

inductive step $i-1 \to i$: $(v_1,\dots,v_{i-1})$ is an orthon. basis of $\text{span}(b_1,\dots,b_{i-1})$

- clearly $y_i \neq 0$ as otherwise $b_1,\dots,b_i$ NOT LI

  $\Rightarrow \|v_i\| = \left\| \dfrac{y_i}{\|y_i\|} \right\| = 1$ again by linearity of a norm
- by 🔔2, $\forall j = 1,\dots,i-1$, $y_i \perp v_j$, thus also $v_i \perp v_j$

  $\Rightarrow$ by $v_1,\dots,v_i$ are mutually orthogonal (and LI).
- It remains to check that $v_1,\dots,v_i$ span $\text{span}(b_1,\dots,b_i)$.

  $\text{span}(b_1,\dots,b_i) \overset{\text{ind. assump.}}{=} \text{span}(v_1,\dots,v_{i-1},b_i) \overset{\text{Exchange lemma on } y_i \text{ (or } v_i\text{) (*)}}{=} \text{span}(v_1,\dots,v_{i-1},v_i)$ ∎

Corollary: Every finite dim VS has ON basis.

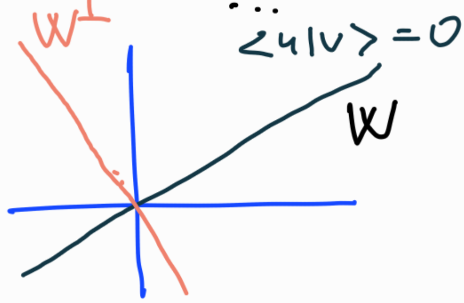
Corollary 2: In a finite dim. VS V: if W is a subspace of V, then every ON basis of W can be extended into an ON basis of V.

Proof: first extend the given ON basis, say $b_1,\dots,b_k$, of W, into a basis $b_1,\dots,b_k,z_1,\dots,z_\ell$ of V by Steinitz. Then Gram-Schmidt.

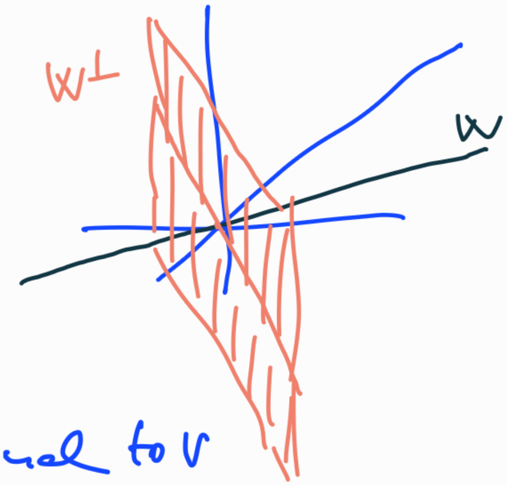
Note: $\forall i = 1,\dots,k$: $v_i = b_i$ .. no change! ∎

Def: Let W be set of vectors from a VS W with inner product. Then the <u>orthogonal complement</u> of W is the set $W^\perp = \{ v \in V : \forall u \in W, u \perp v \}$.
$\vdots$
$\langle u | v \rangle = 0$

E.g. • $V = \mathbb{R}^2$, W ... line through the origin
$W^\perp$ ... perpendicular line through origin



• $V = \mathbb{R}^3$, W ... line through the origin
$W^\perp$ ... plane perpendicular to W, through origin



$U \subseteq V \Rightarrow V^\perp \subseteq U^\perp$

proof: obvious: "everything orthogonal to V is orthogonal to U as well".

$x \in V^\perp \Leftrightarrow \forall v \in V, x \perp v \Rightarrow \forall v \in U, x \perp v \Leftrightarrow x \in U^\perp$ ■

Consider a homogeneous system of lin. eq. $Ax = 0$.
Then $\underset{\{x | Ax = 0\}}{\mathrm{Null}(A)} = R(A)^\perp$. $R(A)$ .. row span of A

proof: $\supseteq$ ... by previous   .... i.e. $z = \sum_i \lambda_i A_{i*}$
$\subseteq$ ... consider $y \in \mathrm{Null}(A)$ and $z \in R(A)$.
We need to check that $\langle z | y \rangle = 0$. $y \in \mathrm{Ker}(A)$

$$\langle \textstyle\sum \lambda_i A_{i*} | y \rangle \overset{L2}{=} \sum \lambda_i \langle A_{i*} | y \rangle = 0$$ ■

Lemma: If $B = (b_1, \dots, b_n)$ is a basis of subspace W, then $v \in W^\perp \Leftrightarrow \forall i = 1, \dots, n,\ v \perp b_i$.

Proof: $\Rightarrow$ obvious
$\Leftarrow$ we have to check that $\forall u \in W$, $\langle u | v \rangle = 0$
$u = \sum_{i=1}^n \alpha_i b_i$ for some α_i. $\langle u | v \rangle = \langle \sum_i \alpha_i b_i | v \rangle \overset{L2}{=} \sum_i \alpha_i \underbrace{\langle b_i | v \rangle}_{=0} = 0$. ■

Theorem (Properties of orthogonal complement).

Let W be a subspace of VS V of finite dimension. Then i) $W^\perp$ is a subspace of V.

ii) $\dim(W) + \dim(W^\perp) = \dim(V)$

iii) $(W^\perp)^\perp = W$

iv) $W^\perp \cap W = \{0\}$.

Proof: i) we check that $W^\perp$ is closed under $+, \cdot$:

- $u, v \in W^\perp$: $\forall x \in W$, $\langle u+v | x\rangle = \underbrace{\langle u|x\rangle}_{=0} + \underbrace{\langle v|x\rangle}_{=0} = 0$, as $u, v \in W^\perp$

$\Rightarrow u+v \in W^\perp$

- $u \in W^\perp$, $\alpha \in T$: $\forall x \in W$, $\langle \alpha u | x\rangle = \alpha \overbrace{\langle u|x\rangle}^{=0} = 0 \Rightarrow \alpha u \in W^\perp$

ii) Let $B = (b_1, \dots, b_k)$ be an ON basis of W. We extend it to an ON basis of V: $H = (b_1, \dots, b_k, c_1, \dots, c_\ell)$.

Now we verify that $C = (c_1, \dots, c_\ell)$ is a basis of $W^\perp$.

Consider any $v \in W^\perp$. Its coordinates w.r.t. H are given by the Fourier coefficients:

$$\left.\begin{array}{l} \overbrace{\langle v | b_i\rangle}^{=0} \dots \text{for the vectors } b_i \\ \langle v | c_j\rangle \dots \text{for the vectors } c_j \end{array}\right\} \Rightarrow v \in \operatorname{span}(C) \Rightarrow C \text{ is basis of } W^\perp.$$

iii) By Lemma, $v \in (W^\perp)^\perp \iff \forall i = 1,\dots,\ell,\ v \perp c_i, \iff$

$\forall i = 1, \dots, \ell$, $\langle v|c_i\rangle = 0 \iff v \in \operatorname{span}(B) = W$.

Fourier coeff. w.r.t. the basis H of V

iv) If $\exists u \in W^\perp \cap W$, $u \neq 0$, then $\exists \alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_\ell$ s.t. $\sum^k \alpha_i b_i = \sum^\ell \beta_j c_j \Rightarrow H$ is not a basis of V - contradiction. ∎

Or: by proof of ii), $W + W^\perp = V$

As $\dim(X+Y) + \dim(X \cap Y) = \dim(X) + \dim(Y)$,

$\dim(W \cap W^\perp) = (k+\ell) - k - \ell = 0$.