BILINEAR & QUADRATIC FORMS

LA 2 | 12 | 5 | 2026 |

Def: A bilinear form on a vector space V, over a field T, is a function $f: V \times V \to T$ satisfying the following two axioms:

(L1) $\forall x, y, z \in V, \forall t \in T: f(x + t \cdot y, z) = f(x,z) + t \cdot f(y,z)$

(L2) —— " —— $: f(x, y + t \cdot z) = f(x,y) + t \cdot f(x,z)$

The bilinear form f is symmetric, if in addition

$\forall x, y \in V, \; f(x,y) = f(y,x)$.

A quadratic form on a vector space V, over a field T, is a function $g: V \to T$ for which there exists a bilinear form f, $\forall x \in V$, $g(x) = f(x,x)$.

Theorem 1: Let V be a vector space over a field T of characteristic not equal 2.

i) Every symmetric bilinear form f on V is uniquely determined by its values $f(u,u)$, $\forall u \in V$.

ii) If g is a quadratic form on V, then there exists a symmetric bilinear form f' s.t. $g(u) = f'(u,u)$, $\forall u \in V$.

Proof: i) Consider any $u, v \in V$. Then

$$f(u+v, u+v) \overset{L1, L2}{=} f(u,u) + f(u,v) + f(v,u) + f(v,v)$$

$$\Rightarrow \overbrace{f(u,v) + f(v,u)}^{*} = f(u+v, u+v) - f(u,u) - f(v,v) \qquad \left(\tfrac{1}{2} = (1+1)^{-1}\right)$$

$$\Rightarrow f(u,v) \overset{\text{symmetry}}{=} \tfrac{1}{2}\left(f(u,v) + f(v,u)\right) = \tfrac{1}{2}\underbrace{\left(f(u+v,u+v) - f(u,u) - f(v,v)\right)}_{\text{only "diagonal" entries}}$$

ii) Let $f'(u,v) = \frac{1}{2}\left(f(u,v) + f(v,u)\right)$ where f determines g.

Then • f' is symmetric • $g(u) = f'(u,u)$, $\forall u \in V$.

Note: in $\mathbb{Z}_2$, $(*)$ equals $2 \cdot f(u,v) = 0$

$\Rightarrow$ we can't express $f(u,v)$

Def: Let V be a VS over T and $B = \{b_1, \dots, b_n\}$ a basis of V.
The matrix of a bilinear form f on V w.r.t the basis B is the matrix A defined by $A_{ij} = f(b_i, b_j)$. "how the basis behaves"

The matrix of a quadratic form g on V w.r.t. the basis B is the matrix of a symmetric bilinear form f s.t. $g(u) = f(u,u)$ $\forall u \in V$, if such f exists.... (think about fields of characteristic 2)

Note 1: Consider a bilinear form f on VS $V = \mathbb{Z}_2^2$ over $\mathbb{Z}_2$ whose matrix w.r.t. the canonical basis is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and a quadratic form $g(u) = f(u,u)$, $\forall u \in V$.
Then there is no symmetric bilinear form f' s.t. $g(u) = f'(u,u)$ $\forall u \in V$.

HW: Why?

Note 2: Given the uniqueness of the symmetric bilin. f for g (THM 1), the definition is correct.

If f is a bilinear form on VS V over T of characteristic other than 2, and A is its matrix w.r.t. a basis $B = \{b_1, \dots, b_n\}$, then the matrix C of the quadratic form $g(u) = f(u,u)$, $\forall u \in V$, w.r.t. the basis B, is given by $C_{ij} = \dfrac{A_{ij} + A_{ji}}{2}$.

Proof: We know from the proof of THM 1, that symmetric bilinear form f' describing g is given by the formula
$$f'(u,v) = \frac{f(u,v) + f(v,u)}{2}, \quad \forall u,v \in V.$$
Thus, $C_{ij} = f'(b_i, b_j) = \dfrac{f(b_i,b_j) + f(b_j,b_i)}{2} = \dfrac{A_{ij} + A_{ji}}{2}$ $\forall i,j$. ∎

Lemma 1: If A is the matrix of a bilin. form f w.r.t. a basis $B = \{b_1, \dots, b_n\}$, then $\forall u,v \in V$: $f(u,v) = [u]_B^T A [v]_B$.

Proof: Let $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_n$ be s.t. $u = \sum_{i=1}^{n} \alpha_i b_i$, $v = \sum_{j=1}^{n} \beta_j b_j$.

Then $f(u,v) = f\left(\sum_{i=1}^{n}\alpha_i b_i, \sum_{j=1}^{n}\beta_j b_j\right) \overset{L1,2}{=} \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \beta_j \underbrace{f(b_i, b_j)}_{A_{ij}} =$

$= [u]_B^T \cdot A [v]_B .$ ∎

Meaning: Every bilinear form (and thus, every quadratic too) can be represented by a matrix.

Conversely, for every square matrix $A \in T^{n\times n}$, VS V over T of dimension n, and a basis B of V, the formula $\forall u,v \in V: f(u,v) = [u]_B^T A [v]_B$ defines a bilin. form on V.

Remark: For a VS V over T, the set of all bilinear forms on V is a group over T ($\sim$ group of all $n\times n$ matrices).

Example: $A = \begin{pmatrix} 1 & 5 \\ 3 & 2 \end{pmatrix}$ $f(x,y) = x^T A y$ ... bilin. form on $\mathbb{R}^2$

$\forall x \in \mathbb{R}^2: g(x) = f(x,x) = x_1^2 + 5x_1x_2 + 3x_2x_1 + 2x_2^2 =$

$= x_1^2 + 4x_1x_2 + 4x_2x_1 + 2x_2^2 = f'(x,x)$

For $A' = \begin{pmatrix} 1 & 4 \\ 4 & 2 \end{pmatrix}$, $f'(x,y) = x^T A' y$ sym. bilin. form defining g.

A is also the matrix of f w.r.t. the canonical basis.

A' —— " —— f' —— " ——.

Note that $f \neq f'$ - e.g. $\overset{5}{\overset{\parallel}{f(e_1,e_2)}} \neq \overset{4}{\overset{\parallel}{f'(e_1,e_2)}}$

Lemma 2: Let A be a matrix of a bilin. form f w.r.t a basis B. Then ${}_B[id]_C^T \cdot A \cdot {}_B[id]_C$ is a matrix of f w.r.t. a basis $C = (c_1, \dots, c_n)$.

(${}_B[id]_C$: transition matrix from C to B)

Proof: We know: $[u]_B = {}_B[id]_C [u]_C$, $[v]_B = {}_B[id]_C [v]_C$

$f(u,v) \overset{\text{Lemma 1}}{=} [u]_B^T \cdot A \cdot [v]_B = [u]_C^T \underbrace{{}_B[id]_C^T \, A \, {}_B[id]_C}_{=: \bar{A}} [v]_C$

$\Rightarrow \forall i,j: f(c_i, c_j) = e_i^T \cdot \bar{A} \cdot e_j = \bar{A}_{ij}$ ∎

Theorem Let f be a symmetric bilinear form on VS V over T of characteristic $\neq 2$. Then there exists a basis B s.t. the matrix of f w.r.t. B is diagonal.

Proof for $T = \mathbb{R}$. Consider any basis B' of V, and let A be the matrix of A w.r.t. B'. Since f is symmetric, A is symmetric, too.

$\Rightarrow$ We know: $\exists$ invertible matrix R s.t. $R^T \cdot A \cdot R$ is diagonal.

Let $B = (b_1, \dots, b_n)$ be a basis of V s.t. ${}_{B'}[id]_B = R$, i.e. $[b_i]_{B'} = R_{*i}$ (↑ i-th column).

Then, by lemma 2, $R^T \cdot A \cdot R = {}_{B'}[id]_B^T \cdot A \cdot {}_{B'}[id]_B$, is a matrix of f w.r.t. B. ☒

Corollary (Sylvester's law of inertia).

For every quadratic form g on VS V of finite dimension over $\mathbb{R}$, there exists a basis B s.t. the matrix of g w.r.t. B is diagonal, only with entries $0, 1, -1$.

Moreover, for each such basis, the numbers of 1 and the number of -1 are invariant.

Proof: Let f be a symmetric bilin. form defining g, i.e., $\forall u \in V$, $g(u) = f(u,u)$.

By the previous Thm, there is a basis B of V s.t. the matrix of f w.r.t. B is diagonal - let D denote it.

Let S and D' are diagonal matrices defined as follows:

for i s.t. $D_{ii} = 0$: $S_{ii} = 1$, $D'_{ii} = 0$

— " — $D_{ii} > 0$: $S_{ii} = \sqrt{D_{ii}}$, $D'_{ii} = 1$

— " — $D_{ii} < 0$: $S_{ii} = \sqrt{-D_{ii}}$, $D'_{ii} = -1$

i.e., $D'_{ii} = \operatorname{sgn} D_{ii}$

Then $D = S^T \cdot D' \cdot S$, and as S is invertible,

$D' = (S^T)^{-1} \cdot D \cdot S^{-1}$ is diagonal, only with entries $0, 1, -1$.

By construction (c.f. lemma 2), D' is a matrix of the quadratic form g, with respect to some basis.

• invariance of the number of +1 and −1 on the diagonal:

Let D and D' be the diagonal matrices of g w.r.t. basis B and C. Then $[id]_{B C}^{T} \cdot D \cdot [id]_{B C} = D'$ ← invertible

$\Rightarrow \text{rank}(D) = \text{rank}(D') \Rightarrow$ the same number of 0's

Assume, wlog, that $B = (b_1, \dots, b_n)$, $C = (c_1, \dots, c_n)$, and also assume, that both D and D' start with the positive elements $+1$ (permut the order in bases)

Let $p \dots \# 1$ in D, $q \dots \# 1$ in D' } For a contradiction, assume $q < p$.

Let $P = \mathcal{L}(\{b_1, \dots, b_p\})$ – linear span of $b_1, \dots, b_p$

$Q = \mathcal{L}(\{c_{q+1}, \dots, c_n\})$.

Since $\dim(P) + \dim(Q) = p + n - q > n$, there exists $v \in P \cap Q$:

$$\Rightarrow \quad v = \sum_{i=1}^{p} \beta_i b_i = \sum_{j=q+1}^{n} \gamma_j c_j \quad , \text{ for some } \beta_i, \gamma_j.$$

Then
$$g(v) = [v]_B^T \cdot D \cdot [v]_B = \sum_{i=1}^{p} \beta_i^2 > 0$$
$$= [v]_C^T \cdot D' \cdot [v]_C = \sum_{j=q+1}^{n} (-1) \cdot \gamma_j^2 \le 0$$
} a contradiction

$\Rightarrow q \ge p$

Symmetrically, we get $p \le q$ } $\Rightarrow p = q$ ☒