Df: A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is

positive definite, if $\forall x \in \mathbb{R}^n, x \neq 0: \; x^T A x > 0$.

positive semidefinite, if $\forall x \in \mathbb{R}^n: \; x^T A x \geq 0$.

Theorem 1: For a symmetric matrix $A \in \mathbb{R}^{n\times n}$, the following conditions are equivalent: i) A is positive definite.

ii) All eigenvalues of A are positive.

iii) $\exists$ invertible matrix U s.t. $A = U^T U$.

iv) $\exists$ upper triangular R with positive diagonal s.t. $A = R^T \cdot R$. — Cholesky decomposition

Corollary: If A is positive definite, then $\det(A) > 0$.

Theorem 2 (Sylvester's criterion of positive definitness):

A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive definite $\iff$ for each $i = 1, \dots, n$, $\det(A_i) > 0$, where A_i is submatrix of A consisting of the first i rows and columns.

Proof: $\Rightarrow$ Note: $\forall i, \forall x \in \mathbb{R}^i, x \neq 0, \; x^T A_i x > 0$, as by the assumption for $\bar{x} = \begin{pmatrix} x \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^n$, $0 < \bar{x}^T A \bar{x} = x^T A_i x$

i.e., $\forall i$, A_i is pos. def. $\overset{\text{Cor.}}{\Longrightarrow} \det(A_i) > 0$

$\Leftarrow$ Consider elementary row operations (EROs) of Gaussian elimination that bring A to REF.

Auxiliary goals: a) show that we only need EROs of the type

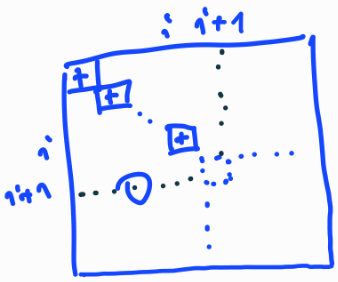
add a t-multiple of row j to row i, $j < i$ $\quad (*)$

b) show that the resulting matrix has positive diagonal, denoted U

we do it by induction on i: number of processed columns:

$\underline{i=1}$: $A_{11}: \det(A_1) > 0 \Rightarrow$ only ERO $(*)$ is needed – a) ✓

$U_{11} = A_{11} > 0$ – b) ✓



$\underline{i \to i+1}$: As ERO $(*)$ does not change determinant:

$0 \underset{\text{Assumption}}{<} \det(A_{i+1}) = \det(\underbrace{U_{i+1}}_{\text{submatrix of } U}) = \underbrace{\prod_{j=1}^{i} U_{jj}}_{>0 \text{ by ind. assump.}} \cdot U_{i+1,i+1} \Rightarrow U_{i+1,i+1} > 0$

$\Rightarrow$ we only need ERO $(*)$ to eliminate column $i+1$

Let E be the lower triangular matrix of the corresponding EROs, with 1's on the diagonal
i.e. $EA = U$.

Let $D = \underbrace{E A \cdot E^T}_{\text{upper}}$. Then $D^T = \underbrace{E \cdot A \cdot E^T}_{\text{lower}}$, i.e., $D = D^T$ & $\Big\}$ D diagonal

As E^T is upper triangular with 1's on the diagonal,
EA and $EAE^T = D$ have the same diagonal
$\Rightarrow$ D has positive diagonal
$\Rightarrow$ $A = E^{-1} D (E^T)^{-1} = R^T R$ for $R = \sqrt{D}(E^T)^{-1}$, i.e., A is pos. def. ∎

Corollary (Gaussian ellimination as a test for PDF).
A symmetric $A \in \mathbb{R}^{n \times n}$ is PDF $\Leftrightarrow$ the sequence of EROs of the type $(*)$ bring it to REF with positive diagonal.
Proof: $\Rightarrow$ in the proof of above.
$\Leftarrow$ For each i, $\det(A_i) = \det(U_i) > 0$, & THM 2. ∎

Two little observations:
1: Let $A, R \in \mathbb{R}^{n \times n}$ be matrices, A symetric, R invertible. Then
A is positive definite $\Leftrightarrow$ $R^T A R$ is PDF.

Proof: e.g. by THM 1, point iii)
A is PDF $\Leftrightarrow$ $\exists$ invertible U s.t. $A = U^T U \Leftrightarrow \exists$ invertible U s.t. $R^T A R = R^T U^T U R$
$\Leftrightarrow$ $R^T A R$ is PDF. ∎

2: Let $A \in \mathbb{R}^{n \times n}$ be a matrix of the form $A = \left[\begin{array}{c|c} \alpha & 0 \\ \hline 0 & B \end{array}\right]$. Then
A is PDF $\Leftrightarrow$ $\alpha > 0$ & B is PDF.

Proof: $\Rightarrow$ Consider $x = (1, 0, \dots, 0)^T$: $0 < x^T A x = \alpha$.
For any $x \in \mathbb{R}^{n-1}$, $x \neq 0$: $x^T B x = (0, x^T) A \binom{0}{x} > 0$.
(and $A = A^T \Rightarrow B = B^T$)
$\Leftarrow$ For any $x \in \mathbb{R}^n$, $x \neq 0$: $x^T A x = x_1^2 \alpha + \bar{x}^T B \bar{x} > 0$ where $\bar{x} = (x_2, \dots, x_n)^T$
$\Rightarrow \underbrace{x_1 \neq 0 \text{ or } \bar{x} \neq 0}$

∎

Theorem (Recursive condition for positive definiteness):

A matrix $A = \left(\begin{array}{c|c} \alpha & a^T \\ \hline a & B \end{array}\right)$ where $\alpha \in \mathbb{R}$, $a \in \mathbb{R}^{n-1}$, $B \in \mathbb{R}^{n-1 \times n-1}$,

is PDF $\iff$ i) $\alpha > 0$, <u>and</u> ii) $B - \frac{a}{\alpha} a^T$ is PDF.

<u>Proof</u>: Let E be a matrix of a sequence of EROs (*) that eliminates the first column below $A_{11} = \alpha$, i.e. $E \cdot A = \left(\begin{array}{c|c} \alpha & a^T \\ \hline 0 & \\ \vdots & ? \\ 0 & \end{array}\right)$

<u>Our question</u>: what will happen with B?

Consider i-th row of A: $(a_{i-1} \quad B_{i-1,*})$ $(i-1)$st row of B

We add to it $\left(-\frac{a_{i-1}}{\alpha}\right)$-multiple of the first row of $A = (\alpha, a^T)$

i.e, altogether, we add $-\frac{a}{\alpha} a^T$ to the matrix B

$$\Rightarrow E \cdot A = \left(\begin{array}{c|c} \alpha & a^T \\ \hline 0 & \\ \vdots & B - \frac{a}{\alpha} a^T \\ 0 & \end{array}\right) \quad \text{for } E = \left(\begin{array}{c|cccc} 1 & 0 & \cdots & & 0 \\ \hline -\frac{a_1}{\alpha} & 1 & & & \\ -\frac{a_2}{\alpha} & 0 & 1 & & 0 \\ \vdots & & & \ddots & \\ -\frac{a_{n-1}}{\alpha} & 0 & \cdots & 0 & 1 \end{array}\right)$$

$$\Rightarrow E \cdot A \cdot E^T = \left(\begin{array}{c|c} \alpha & a^T \\ \hline 0 & \\ \vdots & \underline{B - \frac{a}{\alpha} a^T} \\ 0 & \end{array}\right) \cdot \left(\begin{array}{c|c} 1 & -a^T/\alpha \\ \hline 0 & 1 \quad 0 \\ \vdots & \quad 1 \quad \ddots \\ 0 & \qquad\quad 1 \end{array}\right) = \left(\begin{array}{c|c} \alpha & 0 \cdots 0 \\ \hline 0 & \\ \vdots & \underline{B - \frac{a}{\alpha} a^T} \\ 0 & \end{array}\right)$$

this part does not change!

By 1 and 2: A is PDF $\iff$ EAE^T is PDF $\iff$ $\alpha > 0$ & $B - \frac{a}{\alpha} a^T$ PDF. ∎

Corollary (Computation of Cholesky decomposition by Gauss. el.)

Let $A \in \mathbb{R}^{n\times n}$ be a positive definite matrix and $\bar{U}$ its REF obtained by Gaussian elimination by EROs of the type (*).
Let D be a diagonal matrix s.t. $D_{ii} = \bar{U}_{ii}$, $\forall i = 1, \dots, n$, and let $U = D^{-1} \cdot \bar{U}$. Then $A = U^T D \cdot U$ (and D has positive diagonal.)

Proof: Let E be the matrix of the EROs (*) s.t. $E \cdot A = \bar{U}$.

We know: • $EAE^T = D$ where D is diagonal (proof of Thm 2)

• $\bar{U}$ and D have <u>the same</u> diagonal (positive)

$$A = E^{-1} D \cdot (E^T)^{-1} \Rightarrow (E^T)^{-1} = D^{-1} E \cdot A = D^{-1} \cdot \bar{U} = U \Rightarrow A = U^T \cdot D\, U$$

As D has positive diagonal, we have the Cholesky decomposition: for $R = \sqrt{D} \cdot U$, $A = R^T R$, and R is an upper triangular matrix. ∎

Example:

$$A = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 4 & 7 \\ 2 & 7 & 20 \end{pmatrix}$$

Gaussian elimination:

$$A \sim \begin{pmatrix} 2 & 2 & 2 \\ 0 & 2 & 5 \\ 0 & 5 & 18 \end{pmatrix} \sim \overbrace{\begin{pmatrix} 2 & 2 & 2 \\ 0 & 2 & 5 \\ 0 & 0 & 11/2 \end{pmatrix}}^{\bar{U}} = \overbrace{\begin{pmatrix} 2 & & \\ & 2 & \\ & & 11/2 \end{pmatrix}}^{D} \overbrace{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 5/2 \\ 0 & 0 & 1 \end{pmatrix}}^{U}$$

Verification:

$$U^T D U = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 5/2 & 1 \end{pmatrix}}_{U^T} \underbrace{\begin{pmatrix} 2 & 2 & 2 \\ 0 & 2 & 5 \\ 0 & 0 & 11/2 \end{pmatrix}}_{D \cdot U} = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 4 & 7 \\ 2 & 7 & 20 \end{pmatrix} = A.$$

Def: A <u>bilinear form</u> on a vector space V, over a field T, is a function $f: V \times V \to T$, if it satisfies the following two axioms:

(L1) $\forall x,y,z \in V, \forall t \in T: f(x+t\cdot y, z) = f(x,z) + t\cdot f(y,z)$

(L2) —— " —— $: f(x, y+t\cdot z) = f(x,y) + t\cdot f(x,z)$

The bilinear form f is <u>symmetric</u>, if in addition

$\forall x,y \in V, \ f(x,y) = f(y,x)$.

A <u>quadratic form</u> on a vector space V, over a field T, is a function $g: V \to T$ if for some bilinear form f,

$\forall x \in V, \ g(x) = f(x,x)$.

<u>Theorem:</u> Let V be a vector space over a field T of characteristic not equal 2.

i) Every symmetric bilinear form f on V is <u>uniquely</u> determined by its values $f(u,u)$, $\forall u \in V$.

ii) If g is a quadratic form on V, then there exists a symmetric bilinear form f' s.t. $g(u) = f'(u,u)$, $\forall u \in V$.

<u>Proof:</u> i) Consider any $u, v \in V$. Then

$$f(u+v, u+v) \overset{L1,L2}{=} f(u,u) + f(u,v) + f(v,u) + f(v,v)$$

$$\Rightarrow f(u,v) + f(v,u) = \underbrace{f(u+v,u+v) - f(u,u) - f(v,v)}_{*} \qquad \left(\tfrac{1}{2} = (1+1)^{-1}\right)$$

$$\Rightarrow f(u,v) \overset{\text{symmetry}}{=} \tfrac{1}{2}\left(f(u,v) + f(v,u)\right) = \tfrac{1}{2}\left(f(u+v,u+v) - f(u,u) - f(v,v)\right)$$

ii) Let $f'(u,v) = \frac{1}{2}\left(f(u,v) + f(v,u)\right)$ where f determines g.

Then
- f' is symmetric
- $g(u) = f'(u,u)$, $\forall u \in V$.

$\blacksquare$

Note: in $\mathbb{Z}_2$, $(*)$ equals $2\cdot f(u,v) = 0 \Rightarrow$ can't express $f(u,v)$