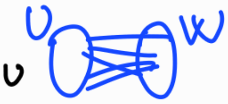
EIGENVALUES & EIGENVECTORS – APPLICATION LA2 21/4/2026

Def: Let $G=(V,E)$ be a graph with $V=\{1,\dots,n\}$. An **adjacency matrix** of G is $A\in\{0,1\}^{n\times n}$ s.t. $A_{ij}=1$ iff $\{i,j\}\in E$.

Theorem: Let G be a graph and $\lambda_1\geq\lambda_2\geq\dots\geq\lambda_n$ eigenvalues of its adjacency matrix. Then G is bipartite $\Leftrightarrow$ $\forall i=1,\dots,n$: $\lambda_i=-\lambda_{n-i+1}$

Proof: $\Rightarrow$ Let U and W be the two parts of G.



The adjacency matrix of G is of the form
$$A=\begin{array}{cc} & \begin{array}{cc} U & W \end{array} \\ \begin{array}{c} U \\ W \end{array} & \left(\begin{array}{c|c} 0 & B \\ \hline B^T & 0 \end{array}\right) \end{array}$$

Let $v=\begin{pmatrix} x \\ y \end{pmatrix}$ be an eigenvector corresponding to eigenvalue λ.

Then $\lambda v=Av$, i.e.,
$$\lambda\begin{pmatrix} x \\ y \end{pmatrix}=\left(\begin{array}{c|c} 0 & B \\ \hline B^T & 0 \end{array}\right)\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} By \\ B^Tx \end{pmatrix}\Rightarrow \begin{array}{l} \lambda y=B^Tx \\ \lambda x=By \end{array}$$

Note:
$$\underbrace{\left(\begin{array}{c|c} 0 & B \\ \hline B^T & 0 \end{array}\right)\begin{pmatrix} x \\ -y \end{pmatrix}}=\begin{pmatrix} -By \\ B^Tx \end{pmatrix}=\begin{pmatrix} -\lambda x \\ \lambda y \end{pmatrix}=\underbrace{-\lambda\begin{pmatrix} x \\ -y \end{pmatrix}},$$

i.e., $\begin{pmatrix} x \\ y \end{pmatrix}$ is an eigenvector to $\lambda \Leftrightarrow \begin{pmatrix} x \\ -y \end{pmatrix}$ is an eigenvector to $-\lambda$.

Moreover, if the geometric multiplicity of λ is k, then the above argument shows that geom. mult. of $-\lambda$ is k too.

$\Leftarrow$ Let k be an odd positive integer, and A the adjacency matrix of G. As A is <u>symmetric</u>, $\exists$ invertible R and diagonal D with $\lambda_1,\dots,\lambda_n$ on the diagonal s.t. $A=R\cdot D\cdot R^{-1}$.

☺1 $A^k=R\cdot D^k\cdot R^{-1}$, i.e., eigenvalues of A^k are $\lambda_1^k,\lambda_2^k,\dots,\lambda_n^k$.

HW: $(A^k)_{ii}>0 \Leftrightarrow \exists$ closed walk of length k from i.

☺2 $\sum_{i=1}^n\lambda_i^k=0$, as $\lambda_i=-\lambda_{n-i+1}$ and k is odd.

<u>We know</u>: the sum of eigenvalues equals the sum of the main diagonal $\Rightarrow \sum_{i=1}^n (A^k)_{ii}=\sum_{i=1}^n\lambda_i^k=0$

As $(A^k)_{ii}\geq 0, \forall i \Rightarrow \forall i, (A^k)_{ii}=0$.

<u>If</u> there is an odd cycle C of length k, then $\forall i\in C$, $(A^k)_{ii}\geq 1$

$\Rightarrow$ no odd cycle in G $\Rightarrow$ G is bipartite. ∎

JORDAN NORMAL FORM

We know that some matrices are NOT diagonalizable, e.g. $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Our question: how close to a diagonal matrix can we get?

Def: A <u>Jordan block</u> $J_k(\lambda)$ of <u>order k</u> is a $k \times k$ matrix

$$J_k(\lambda) = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$$

i.e., all entries on the main diagonal are λ
all entries immediately above it are 1
all other entries are 0

The only eigenvalue of $J_k(\lambda)$ is λ.

<u>proof</u>: $J_k(\lambda)$ is an upper triangular $\to$ eigenvalues are on the main diagonal. $\square$

Def: A matrix J is in Jordan normal form if it is of the form

$$J = \begin{pmatrix} \boxed{J_1} & & & & 0 \\ & \boxed{J_2} & & & \\ & & \boxed{J_3} & & \\ & & & \ddots & \\ 0 & & & & \boxed{J_m} \end{pmatrix}$$

where each J_i is a Jordan block $J_{k_i}(\lambda_i)$ for some k_i, λ_i.

The λ_i's <u>need not</u> be pairwise distinct.

E.g. $\begin{pmatrix} 5 & 1 & & 0 \\ & 5 & 0 & \\ & & 2 & 0 \\ 0 & & & 5 \end{pmatrix}$, $\begin{pmatrix} 2 & 1 & & 0 \\ & 2 & & \\ & & 3 & 1 \\ 0 & & & 3 \end{pmatrix}$, $\begin{pmatrix} 1 & & 0 \\ & 2 & \\ 0 & & 3 \end{pmatrix}$

<u>Theorem</u> (Jordan normal form):

Every matrix $A \in \mathbb{C}^{n \times n}$ is similar to a matrix J in Jordan normal form. The matrix J is unique, up to the order of the blocks.

Without proof.

POSITIVE DEFINITE MATRICES

Df: A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is

positive definite, if $\forall x \in \mathbb{R}^n, x \neq 0: x^T A x > 0$.

positive semidefinite, if $\forall x \in \mathbb{R}^n: x^T A x \geq 0$.

Theorem (Characterization of positive definite matrices):

For a symmetric matrix $A \in \mathbb{R}^{n \times n}$, the following conditions are equivalent.

i) A is positive definite.

ii) All eigenvalues of A are positive.

iii) $\exists$ invertible matrix U s.t. $A = U^T U$.

iv) $\exists$ upper triangular matrix R with positive diagonal s.t. $A = R^T \cdot R$. Cholesky decomposition

Proof: i) $\Rightarrow$ ii)

Consider an eigenvalue λ of A. Then $Au = \lambda u$ for some $u \in \mathbb{R}^n$.

$0 < u^T A u = u^T \lambda u = \lambda \underbrace{u^T u}_{>0} \Rightarrow \lambda > 0$ $\square$

ii) $\Rightarrow$ iii) We know: symmetric matrices are diagonalizable by orthogonal matrices: $\exists$ orthogonal B & diagonal D s.t. $A = B^{-1} D B = B^T D B$ & D_{ii}'s are the eigenvalues of A

$\Rightarrow$ for $U = \sqrt{D} \cdot B$, $A = U^T \cdot U$, where $(\sqrt{D})_{ii} = \sqrt{D_{ii}}$ $\square$

iii) $\Rightarrow$ iv)

We know: Each invertible matrix U is decomposable in an orthogonal Q and upper triangular R with positive diagonal:

$U = Q \cdot R \Rightarrow A = U^T U = R^T \underbrace{Q^T Q}_{= I} \cdot R = R^T \cdot R$. $\square$

iv) $\Rightarrow$ i) $x^T A x = x^T R^T \underbrace{Rx}_{\neq 0} > 0$

$\neq 0$ ... R - positive diag. - invertible $\square$

Corollary: If A is positive definite, then $\det(A) > 0$.

Proof: We know: det is the product of eigenvalues.

Note 1: Analogous theorem holds for positive semidefinit.

Note 2: Positive definite matrices important in OPTIMIZATION (e.g., for MAXCUT).

Note 3: The matrix R from iv) is unique.

proof: $A = R^T R = P^T P$, R, P - upper triang. with positive diagonal

$$\underbrace{\underbrace{R P^{-1}}_{\text{both upper } \Delta}}_{\text{upper } \Delta} = \underbrace{\underbrace{(R^T)^{-1} P^T}_{\text{both lower } \Delta}}_{\text{lower } \Delta} \Rightarrow \underline{\text{diagonal}}$$

We have: $RP^{-1} = D = (R^T)^{-1} P^T$ D is symmetric

i.e. $R = D \cdot P$

$$D^{-1} = (P^T)^{-1} \cdot R^T = RP^{-1}$$

$\Rightarrow D = D^{-1}$, i.e., the diagonal contains only ± 1

If $\exists i$ s.t. $D_{ii} = -1$, then $R_{ii} = -P_{ii}$ - contradiction with the assumption that both P, R have positive diagonals

$\Longrightarrow \forall i, D_{ii} = 1$, i.e., $D = I$. Thus $\underline{R} = DP = \underline{P}$. ∎

Example: For a basis $(b_1, \dots, b_n)$ of VS V over $\mathbb{R}$ with inner product, the Gram matrix $G \in \mathbb{R}^{n \times n}$ is defined by $G_{ij} := \langle b_i | b_j \rangle$. $\forall i,j$.

☺ G is positive definite.

Proof: for $x \neq 0$: $x^T G x = \sum_{i=1}^{n} \sum_{j=1}^{n} G_{ij} x_i x_j = \sum_{i=1}^{n} \sum_{j=1}^{n} \langle b_i | b_j \rangle x_i x_j \overset{\text{linearity of inner product}}{=}$

$= \sum_{i=1}^{n} \sum_{j=1}^{n} \langle x_i b_i | x_j b_j \rangle = \sum_{i=1}^{n} \langle x_i b_i | \sum_{j=1}^{n} x_j b_j \rangle = \langle \overbrace{\sum_{i=1}^{n} x_i b_i} | \overbrace{\sum_{j=1}^{n} x_j b_j} \rangle > 0$ ∎

non-trivial lin. comb. of basis

☺ If A is positive definite matrix, then $\langle x | y \rangle = x^T A y$ defines inner product. - obvious

$\Rightarrow$ close connection between inner product & positive def. matrices