INNER PRODUCT SPACES

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Most of you know the Euclidean space:

- vectors (n-tuples from $\mathbb{R}^n$) plus lenght (distance) and angle
- (standard) dot product of vectors:

$$\langle x|y\rangle = x_1y_1 + x_2y_2 + \dots + x_ny_n = x^Ty \dots \text{matrix multiplic.}$$

- lenght of a vector x: $\|x\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} = \sqrt{\langle x|x\rangle}$
- angle of vectors x, y: $\cos\varphi = \dfrac{\langle x|y\rangle}{\|x\|\cdot\|y\|}$

Note: • both lenght and angle derived from the dot product!

- if y is fixed, then $\langle\cdot\cdot|y\rangle$ (i.e., the dot product with y) is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}$

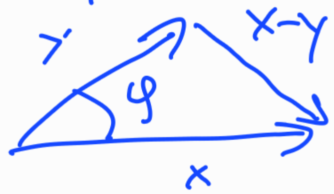
$\Rightarrow$ a possible view of $\langle x|y\rangle$:

x ... a static vector that is being manipulated with

y ... an active vector (matrix) that does something with x

the roles can be switched as $\langle x|y\rangle = \langle y|x\rangle$

- $\langle x-y|x-y\rangle \overset{\text{linearity}}{=} \langle x|x\rangle - 2\langle x|y\rangle + \langle y|y\rangle$



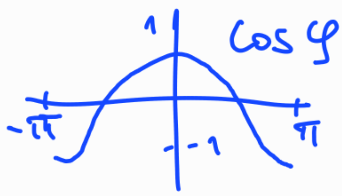
i.e., $\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\cdot\|y\|\cdot\cos\varphi$.

. the Law of Cosine.

- for perpendicular vectors - $\cos\varphi = 0 \Rightarrow$

$\|x-y\|^2 = \|x\|^2 + \|y\|^2$ - Pythagorean theorem

- for unit vectors $\langle x|y\rangle = \cos\varphi$ -



similarity measure

important for data analysis

Our plan: define inner product in general for VS over $\mathbb{R}$ and $\mathbb{C}$

Recall: for a complex number $z = a + i\cdot b$: $\underbrace{\bar{z}}_{\text{complex conjugate}} = a - i\cdot b$

Def: Let V be a VS over $\mathbb{R}$ (or $\mathbb{C}$). Then a mapping $V \times V \to \mathbb{R}$ (or $\to \mathbb{C}$) that assigns to a pair x, y of vectors from V a real (or complex) number, denoted $\langle x|y\rangle$ is called an **inner** (scalar) **product** if it satisfies the following axioms:

linear
$$\begin{cases} \text{(L1)} \ \forall x,y \in V, \ \forall \alpha \in \mathbb{R} \ (\text{or } \mathbb{C}): \langle \alpha x|y\rangle = \alpha \langle x|y\rangle \\ \text{(L2)} \ \forall x,y,z \in V: \quad \langle x+y|z\rangle = \langle x|z\rangle + \langle y|z\rangle \end{cases}$$

positive → (P) $\forall x \in V$: $\quad \langle x|x\rangle \geq 0$ and $\langle x|x\rangle = 0$ only for $x = 0$

commut. → (C) $\forall x,y \in V$: $\quad \langle x|y\rangle = \underbrace{\overline{\langle y,x\rangle}}_{\text{complex conjugate}}$ (i.e. $\langle x|y\rangle = \langle y,x\rangle$ for $\mathbb{R}$)

Note: • (C) implies: $\langle x|x\rangle = \overline{\langle x|x\rangle}$, $\forall x \in V$ over $\mathbb{C}$, i.e., $\langle x|x\rangle \in \mathbb{R}$!

(HW)

• $\forall x,y,z \in V$: $\langle x|y+z\rangle = \langle x|y\rangle + \langle x|z\rangle$ (by (C), (L1), (C))

• $\forall x,y \in V$, $\forall \alpha \in \mathbb{C}$ $\quad \langle x|\alpha y\rangle = \bar{\alpha}\langle x|y\rangle$ .... (L1')

• $\forall x \in V$: $\underline{\langle x|0\rangle = 0}$ (as $\langle x|0 \cdot x\rangle = 0 \cdot \langle x|x\rangle = 0$)

Examples:

HW: check the axioms

• $\mathbb{R}^2$: $\langle x|y\rangle = x_1 y_1 + 2x_1 y_2 + 2x_2 y_1 + 5x_2 y_2$

• For an invertible matrix A: $\langle x|y\rangle = x^T A^T A y$

• (standard) dot product on $\mathbb{C}^n$:

$$\langle x|y\rangle = \sum_{i=1}^{n} x_i \cdot \overline{y_i} \quad \left(= y^H \cdot x \quad \text{hermit transpose}\right)$$

Non-example! $x^T y$ in $\mathbb{C}^n$ is NOT an inner product

(P) does not hold!

Def: Let V be a VS over $\mathbb{R}$ (or $\mathbb{C}$). A mapping $V \to \mathbb{R}$ (always to $\mathbb{R}$), that assignes to a vector $x \in V$ a number $\|x\|$ is called a **norm** if it satisfies the following axioms:

(P) $\forall x \in V: \ \|x\| \geq 0$, and $\|x\| = 0$ only for $x = 0$

(L) $\forall x \in V, \forall \alpha \in \mathbb{R}$ (or $\mathbb{C}$): $\ \|\alpha x\| = |\alpha| \cdot \|x\|$

(TI) $\forall x, y \in V: \ \|x + y\| \leq \|x\| + \|y\|$

HW: Verify that $\|x\|_2 := \sqrt{\sum_{i=1}^{n} x_i^2}$, $\|x\|_1 := \sum_{i=1}^{n} |x_i|$, and $\|x\|_\infty := \max\{|x_1|, \dots, |x_n|\}$ are norms in $V = \mathbb{R}^n$.

Def: A norm induced by an inner product: a mapping that assignes the number $\|x\| = \sqrt{\langle x | x \rangle}$ to a vector x.

Theorem (Cauchy-Schwarz ineq.): If V is a VS over $\mathbb{R}$ (or $\mathbb{C}$) with an inner product, and $\|x\|$ is a norm induced by the inner product, then

$$\forall x, y \in V: \ |\langle x | y \rangle| \leq \|x\| \cdot \|y\|.$$

Proof: For $\mathbb{R}$: If $x = 0$ or $y = 0$, then obvious.

Assume $x \neq 0$ and $y \neq 0$. Let $z = \frac{x}{\|x\|} \pm \frac{y}{\|y\|}$.

Then $0 \leq \langle z | z \rangle = \left\langle \frac{x}{\|x\|} \pm \frac{y}{\|y\|} \,\middle|\, \frac{x}{\|x\|} \pm \frac{y}{\|y\|} \right\rangle =$

$$= \underbrace{\frac{\langle x | x \rangle}{\|x\|^2}}_{=1} \pm \frac{\langle x | y \rangle}{\|x\| \cdot \|y\|} \pm \frac{\langle y | x \rangle}{\|y\| \cdot \|x\|} + \underbrace{\frac{\langle y | y \rangle}{\|y\|^2}}_{=1} = 2 \pm 2 \frac{\langle x | y \rangle}{\|x\| \cdot \|y\|}$$

$$\left.\begin{array}{l} \Rightarrow \ \langle x | y \rangle \leq \|x\| \cdot \|y\| \\ \Rightarrow \ -\langle x | y \rangle \leq \|x\| \cdot \|y\| \end{array}\right\} \Rightarrow \ |\langle x | y \rangle| \leq \|x\| \cdot \|y\|$$

$\square$

For $\mathbb{C}$: For $y = 0$ obvious. Wlog assume $\|y\| = 1$
:HW

$$0 \le \langle x - \langle x|y\rangle y \,|\, x - \langle x|y\rangle y\rangle \overset{L2}{=}$$

$$= \langle x | x - \langle x|y\rangle y\rangle - \langle x|y\rangle \underbrace{\langle y | x - \langle x|y\rangle y\rangle}_{=0} \overset{L2, L1'}{=}$$

$$= \langle x|x\rangle - \underbrace{\overline{\langle x|y\rangle}\langle x|y\rangle}_{=|\langle x|y\rangle|^2} - \langle x|y\rangle\langle y|x\rangle + \langle x|y\rangle\overline{\langle x|y\rangle}\underbrace{\langle y|y\rangle}_{=1}$$

$$= |\langle x|y\rangle|^2 \quad \Rightarrow |\langle x|y\rangle| \le \|x\| \cdot \underbrace{\|y\|}_{=1} \qquad \blacksquare$$

recall: for $z = a+bi$, $|z| = \sqrt{a^2+b^2}$;
$z \cdot \bar{z} = (a+bi)(a-bi) = a^2+b^2 = |z|^2$

Corollary: In a VS V over $\mathbb{R}$ (or $\mathbb{C}$), the norm induced by an inner product, is a norm.

Proof: (P) of $\|x\|$ follows from (P) of $\langle x|x\rangle$

(L): $\langle \alpha x | \alpha x\rangle \overset{L1}{=} \alpha\langle x|\alpha x\rangle \overset{L1'}{=} \alpha\cdot\bar{\alpha}\langle x|x\rangle = |\alpha|^2 \cdot \langle x|x\rangle$

i.e., $\|\alpha x\|^2 = |\alpha|^2 \cdot \|x\|^2 \Rightarrow \|\alpha x\| = |\alpha| \cdot \|x\|$ ✓

(TI): $\|x+y\| = \sqrt{\langle x+y|x+y\rangle} = \sqrt{\langle x|x\rangle + \underbrace{\langle x|y\rangle + \langle y|x\rangle}_{\overset{(C)}{=} \langle x|y\rangle + \overline{\langle x|y\rangle} \le 2|\langle x|y\rangle|} + \langle y|y\rangle}$

$z + \bar{z} \le 2|z|$

Cauchy-Schwarz

$$\le \sqrt{\|x\|^2 + 2\|x\|\cdot\|y\| + \|y\|^2} = \|x\| + \|y\|. \qquad \blacksquare$$

ORTHOGONALITY

Def: Vectors x, y in a VS V with an inner product $\langle\cdot|\cdot\rangle$ are orthogonal, if $\langle x|y\rangle = 0$.

Notation: $x \perp y$

Every set of non-zero mutually orthogonal vectors is linearl independent.

Corollary (Arithmetic & Quadratic mean (RMS))

For every $x \in \mathbb{R}^n$, $\frac{1}{n} \cdot \sum_{i=1}^{n} x_i \leq \sqrt{\frac{1}{n} \cdot \sum_{i=1}^{n} x_i^2}$.

Proof: Let $y = (1, \dots, 1)^T$. HW?

Then $\langle x | y \rangle = \sum_{i=1}^{n} x_i$ $\quad \|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$ $\quad \|y\| = \sqrt{n}$

Cauchy-Schwarz

$$\sum_{i=1}^{n} x_i \leq \left| \sum_{i=1}^{n} x_i \right| \overset{\text{Cauchy-Schwarz}}{\leq} \sqrt{n \cdot \sum_{i=1}^{n} x_i^2} \quad / \text{ divide by } n$$

$$\Rightarrow \quad \frac{1}{n} \sum x_i \leq \sqrt{\frac{1}{n} \sum x_i^2}$$

∎