VECTOR SPACES, contd.

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Key notions: linear combination (LC) of vectors $v_1, \dots, v_k$

linear span of a collection U of vectors - $\mathrm{span}(U)$

subspace of a VS (set closed under $+, \cdot$)

THE FOUR FUNDAMENTAL SUBSPACES

Def: For a field T and a matrix $A \in T^{m \times n}$ we define:

1. the column space of A: $C(A) = \mathrm{span}(\{A_{*1}, \dots, A_{*n}\})$ Col(A)
2. the row space of A: $R(A) = C(A^T)$ Row(A)
3. the null space of A: $N(A) = \{x \in T^n : Ax = 0\}$ a.k.a. kernel, Null(A)
4. the null space of A^T: $N(A^T)$

Note: we already know that all of these are subspaces.

Theorem: Let $A \in T^{m \times n}$, and let $R \in T^{m \times m}$ be invertible.
Then $R(A) = R(R \cdot A)$ and $N(A) = N(R \cdot A)$.
("left multiplication by an invertible matrix does not change the row and null spaces")

Proof: we know: every row of $R \cdot A$ is a LC of rows of A

$$\boxed{\begin{matrix} R \\ \cdots \end{matrix}} \; \boxed{A} = \boxed{\begin{matrix} R\cdot A \\ \cdots \end{matrix}}$$

$\Rightarrow$ LC of rows of $R \cdot A$ is a LC of rows of A

$\Rightarrow \mathrm{span}(\text{rows of } R\cdot A) \subseteq \mathrm{span}(\text{rows of } A)$, i.e., $R(R\cdot A) \subseteq R(A)$

As R is invertible, there exists R^{-1} s.t. $R \cdot R^{-1} = I$.

Thus, $A = \underset{\uparrow \text{ invertible}}{R^{-1}} \cdot (R \cdot A)$, and by the above argument: $R(A) \subseteq R(R \cdot A)$

$\Rightarrow R(A) = R(R \cdot A)$.

Null space: consider $x \in N(A)$, i.e., $Ax = 0$.
Then $R \cdot A \cdot x = R \cdot 0 = 0$, i.e., $x \in N(R \cdot A)$
Consider $x \in N(RA)$, i.e. $R \cdot A \cdot x = 0$
Then $A \cdot x = R^{-1} \cdot (R \cdot A) x = R^{-1} \cdot 0 = 0$, i.e., $x \in N(A)$

$\Big\} \; N(A) = N(RA)$

Corollary: EROs do not change $R(A)$ and $N(A)$.

Def: A collection (finite sequence) of vectors $v_1, \dots, v_n$ is <u>linearly independent</u> (LI) if only the trivial linear combination of these vectors equals the zero vector,
(all coef. zero)
i.e., if $c_1 v_1 + \dots + c_n v_n = \vec{0} \Rightarrow c_1 = \dots = c_n = 0$.
Otherwise they are <u>linearly dependent</u> (LD)

E.g.: $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$: $1 \cdot v_1 + 1 \cdot v_2 + (-4) v_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$ LD

<u>Note</u>:
- if $v_i = v_j$ for some $i \neq j \Rightarrow$ LD $c_i = 1$, $c_j = -1$, all other $= 0$
- if $v_i = \vec{0}$ for some $i \Rightarrow$ LD $c_i = 1$, all other $= 0$
- if v_n is linear combination of $v_1, \dots, v_{n-1} \Rightarrow$ LD
- if $v_n \in \operatorname{span}(\{v_1, \dots, v_{n-1}\}) \Rightarrow$ LD

<u>Checking whether</u> $v_1, \dots, v_n$ are LI/LD:
- solve $Ax = 0$ for $A = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$: only 0 solution - LI, otherwise LD
- convert $A = \begin{pmatrix} - v_1^T - \\ - v_n^T - \end{pmatrix}$ to REF: if there is a null row $\to$
  $\to$ vectors are LD (otherwise LI. HW: Why?)

<u>Observation</u>: If $v_1, \dots, v_n$ are LD, then $\exists i \in \{1, \dots, n\}$ and coefficients $b_j \in T$, for $j \neq i$, s.t. $v_i = \sum_{j \neq i} b_j v_j$.

<u>Proof</u>: There exist $a_1, \dots, a_n$, not all zero, s.t. $\sum_{j=1}^{n} a_j \cdot v_j = \vec{0}$. Let i be such that $a_i \neq 0$.
By adding $\left(-\sum_{j \neq i} a_j v_j\right)$ to both sides, we get

$$a_i v_i = -\sum_{j \neq i} a_j v_j.$$ Multiply both sides by a_i^{-1}:

$$\Rightarrow \quad v_i = (a_i^{-1}) \cdot \sum_{j \neq i} a_j v_j = \sum_{j \neq i} (a_i^{-1} \cdot a_j) \cdot v_j. \quad \blacksquare$$

<u>Corollary</u>: $v_1, \dots, v_n$ are LI $\iff$ HW
$\iff \forall i,\ v_i \notin \operatorname{span}(\{v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n\})$.

Def: A set B of vectors from a VS V is a basis of V if i) B is linearly independent, and
ii) $\mathrm{span}(B) = V$.

Ex.: $e_1, \dots, e_n$, where $e_i = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \leftarrow$ i-th, is a basis of $\mathbb{R}^n$ over $\mathbb{R}$.

Does every VS have a basis?

Observation: Let X be a collection of vectors from VS V s.t. i) $\mathrm{span}(X) = V$

"basis - minimal spanning set"

ii) $\forall v \in X$, $\mathrm{span}(X \setminus \{v\}) \neq V$.

Then X is a basis of V.

Proof: we need to check that X is LI.

$\therefore$ implies $\forall v \in X$, $v \notin \mathrm{span}(X \setminus \{v\})$.

By previous Corollary, this means X is LI. ☒

Def: A collection U of vectors from VS V is a spanning collection (SC), if $\mathrm{span}(U) = V$.

Corollary: Any finite spanning collection of V can be reduced to a basis by discarding some vectors.

Proof: Consider a spanning set $X = \{v_1, \dots, v_n\}$ of V.

If exists i s.t. $\mathrm{span}(X \setminus \{v_i\}) = \mathrm{span}(X)$, remove v_i from the spanning set $\rightsquigarrow$ a smaller set.

If it does not exist, X is LI $\Rightarrow X$ is a basis. ☒

Corollary 3.3

Theorem: Every VS has a basis.

For finitely generated — by the above Corollary, for other — without a proof.

Can bases of a given VS have different sizes?

Lemma (exchange): Let $v_1, \dots, v_n$ be a spanning collection (SC) of VS V over T, and $w = \sum_{i=1}^{n} a_i v_i$ for some $a_1, \dots, a_n \in T$.
Then for each i s.t. $a_i \neq 0$, $v_1, \dots, v_{i-1}, w, v_{i+1}, \dots, v_n$ is also a spanning collection of V.

Example: $V = \mathbb{R}^3$, (e_1, e_2, e_3) is spanning collection of V.
For $w = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = 2 \cdot e_1 + 3 \cdot e_2$, we have $a_1 \neq 0$, $a_2 \neq 0$,
Thus, (w, e_2, e_3) and (e_1, w, e_3) are SCs of $\mathbb{R}^3$.

$e_1 = \frac{1}{2}(w - 3e_2)$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \cdot e_1 + y \cdot e_2 + z \cdot e_3 = x\left(\frac{1}{2} w - \frac{3}{2} e_2\right) + y \cdot e_2 + z \cdot e_3 =$$

$$= \frac{x}{2} \cdot w + \left(y - \frac{3}{2}x\right) \cdot e_2 + z \cdot e_3 \quad \dots$$

i.e., every $(x, y, z)^T \in \mathbb{R}^3$ is a LC of w, e_2, e_3

Proof: Consider i s.t. $a_i \neq 0$.

Then $v_i = a_i^{-1} \cdot \left(w - \sum_{j=1, j \neq i}^{n} a_j v_j\right)$. (*)

Consider arbitrary $z \in V$. By the assumptions, there exists $b_1, \dots, b_n \in T$ s.t. $z = \sum_{j=1}^{n} b_j v_j$

Substitute for v_i by (*):

$$z = \sum_{j=1, j \neq i}^{n} b_j \cdot v_j + b_i \cdot v_i = \underbrace{\sum_{j=1, j \neq i}^{n} (b_j - b_i a_i^{-1} a_j) v_j + a_i^{-1} b_i w}_{\text{LC of}}$$

$\Rightarrow$ it is a SC.

- Gaussian elimination - zero row in REF $\Rightarrow$
  $\Rightarrow$ original rows are LD

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Theorem (Steinitz exchange lemma): If $I = (w_1, \dots, w_m)$ are linearly independent vectors and $S = (v_1, \dots, v_n)$ is a spanning collection of a VS V, then $m \leq n$, and I can be extended by $n-m$ vectors from S into a spanning collection of V.

Corollary: Every two bases of a finitely generated VS V have the same size.

Def: The dimension of a finitely generated VS V is the size of any of its bases. Notation: $\dim(V)$.