VECTOR SPACES

20/11/2025

Review: A vector space (VS) over a field $(T, +, \cdot)$ – a set V with a binary operation $+$ on V and operation $\cdot : T \times V \to V$ s.t.:

1. $(V, +)$ is Abelian group; neutral element 0, inverse $-u$
2. $\forall a, b \in T, \forall v \in V: a \cdot (b \cdot v) = (a \cdot b) \cdot v$
3. $\forall v \in V: 1 \cdot v = v$ where 1 is neutral element in T for $\cdot$
4. $\forall a, b \in T, \forall v \in V: (a+b) \cdot v = a \cdot v + b \cdot v$
5. $\forall a \in T, \forall u, v \in V: a \cdot (u+v) = a \cdot u + a \cdot v$

distributivity

E.g. $V = \mathbb{R}^2$ over $\mathbb{R}$ $\quad$ $+$ ... component-wise, $\cdot$ ...

Theorem (basic properties of VS): Let $(V, +, \cdot)$ be a vector space over a field T. Then

1. $\forall a \in T: \; a \cdot \vec{0} = \vec{0}$
2. $\forall u \in V: \; \underset{\in T}{0} \cdot u = \underset{\in V}{\vec{0}}$
3. $a \cdot u = \vec{0} \iff a = 0^{\in T}$ or $u = \vec{0}^{\in V}$.

Proof: 1. As $(V, +)$ is a group, $\exists y \in V$ (inverse elem. for $a \cdot \vec{0}$) s.t. $a \cdot \vec{0} + y = \vec{0}$.
Thus, $\underline{a \cdot \vec{0}} = a \cdot \vec{0} + \underbrace{a \cdot \vec{0} + y}_{= \vec{0}} = a \cdot (\vec{0} + \vec{0}) + y = a \cdot \vec{0} + y = \underline{\vec{0}}$

2. Similarly, $\exists y \in V$ s.t. $0 \cdot u + y = \vec{0}$
$\Rightarrow \underline{0 \cdot u} = 0 \cdot u + 0 \cdot u + y = 0 \cdot u + y = \underline{\vec{0}}$

3. By a contradiction, assume $\exists a \in T, u \in V$ s.t.
$a \cdot u = \vec{0}$ & $a \neq 0$ & $u \neq \vec{0}$.
Then $\underline{u} = 1 \cdot u = (a^{-1} \cdot a) \cdot u = a^{-1} \cdot \vec{0} = \underline{\vec{0}}$ (by point 1) – a contradiction. ∎

Def: Let $(V,+,\cdot)$ be a vector space over T and let $\emptyset \neq U \subseteq V$ s.t. i) $\forall u,v \in U: u+v \in U$
ii) $\forall a \in T, \forall u \in U: a \cdot u \in U$. } i.e., U is closed under $+$ and $\cdot$

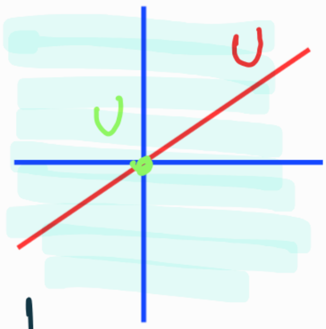
non-empty!

Then $(U,+,\cdot)$ is a (vector) subspace of V, where $+$ and $\cdot$ denote the inherited operations from V.

Every subspace $(U,+,\cdot)$ of a VS V over T is also a vector space over T. HW - check the axioms

Examples: For $V=\mathbb{R}^2$, the subspaces are

- the origin $U=\{(0,0)^T\} \subseteq \mathbb{R}^2$ non-empty closed
- every line passing through the origin
- entire $\mathbb{R}^2$



Note: the three kinds of subspaces differ - dimensions $0,1,2$ - a notion to be defined | non-examples

Def: A linear combination (LC) of vectors $v_1,\dots,v_k \in V$ over T is any sum $a_1v_1+\dots+a_kv_k$ where $a_1,\dots,a_k \in T$ (always a finite number of vectors).

We define the LC of empty set of vectors to be $\vec{0}$ - the zero vector of V.

Note: LC are everywhere!

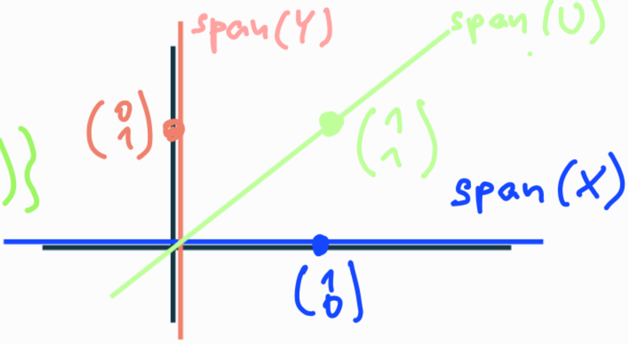
Def: Let V be a vector space over T and X be subset of vectors from V. The linear span (or simply span) $\mathrm{span}(X)$ of X is the set of all linear combinations of vectors in X, i.e.,

$$\mathrm{span}(X)=\{a_1v_1+\dots+a_kv_k : k \in \mathbb{N}_0, v_1,\dots,v_k \in X, a_1,\dots,a_k \in T\}.$$

Ex. $V=\mathbb{R}^2$ (over $\mathbb{R}$)

$X=\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}$ $Y=\left\{\begin{pmatrix}0\\1\end{pmatrix}\right\}$ $U=\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$

$Z=\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$ $\mathrm{span}(Z)=\mathbb{R}^2$



Theorem: Let V be a vector space over T and $X \subseteq V$. Then span(X) is a subspace of V. (*)

Proof: • is span$(X) \neq \emptyset$? - non-empty?

For every $X \subseteq V$ (even for $X = \emptyset$), $\overline{0} \in$ span(X) ✓

• is span$(X) \subseteq V$? all LC of vectors from X are in V (application of $+, \cdot$)

• is span(X) closed under $+$ and $\cdot$?

consider $u, v \in$ span(X), $t \in T$. u, v are LC of vectors from X

Then $u = a_1 u_1 + \dots + a_k u_k$, for some $k \in \mathbb{N}, u_i \in X, a_i \in T$

$v = b_1 v_1 + \dots + b_\ell v_\ell$, — " — $\ell \in \mathbb{N}, v_i \in X, b_i \in T$

addition: $u + v = a_1 u_1 + \dots + a_k u_k + b_1 v_1 + \dots + b_\ell v_\ell$

thus, $u+v$ is LC of $u_1, \dots, u_k, v_1, \dots, v_\ell$, $u+v \in$ span(X).

scalar multiplication - $t \in T$:

$$t \cdot u = t \cdot (a_1 u_1 + \dots + a_k u_k) \overset{\text{distrib. property}}{=} (t \cdot a_1) u_1 + \dots + (t a_k) u_k$$

- thus $t \cdot u$ is LC of $u_1, \dots, u_k$, $t \cdot u \in$ span(X). ∎

Theorem (intersection of subspaces). Let U_i, $i \in I$, (possibly infinite I) be subspaces of a VS V over T. Then $U = \bigcap_{i \in I} U_i$ is also a subspace of V.

Proof: • as $\forall i \in I$, $\overline{0} \in U_i$, $\overline{0} \in U$, i.e., $U \neq \emptyset$. non-empty

$U_i \subseteq V \Rightarrow U \subseteq V$

• closed under $+, \cdot$? Consider $u, v \in U$, $t \in T$.

Then $\forall i \in I, u, v \in U_i \Rightarrow \forall i \in I, u+v \in U_i \Rightarrow u+v \in U$

$\forall i \in I, t \cdot u \in U_i \Rightarrow t \cdot u \in U$ ∎

Eg. for a homogenous system $Ax = 0$ of linear equations,

solutions of i-th equation - subspace U_i

solution of $Ax = 0$ - intersection of all U_i's

Theorem (description of span(X) from outside).
Let V be a VS over $\mathbb{T}$ and $X \subseteq V$. Then span(X) is $\underbrace{\text{the intersection of all subspaces of } V \text{ containing } X}_{=W}$.

Note: definition of span(X) - description from inside

Proof: We need to show span$(X) = W$ ... $\subseteq$ $\supseteq$

- by the previous THM, W is a subspace of V
- by definition, $X \subseteq W$
- as every subspace is closed under $+$ and $\cdot$,
  span$(X) \subseteq W$ (span(x) are LC of X)

- by THM ⊛, span(X) is a subspace
- by definition, $X \subseteq$ span(X)
- span(X) is one of the intersected subspaces

$\Rightarrow W \subseteq$ span(X). ∎

Def: A collection (finite sequence) of vectors $v_1, \dots, v_n$ is linearly independent (LI) if only the trivial linear ("all coef. zero") combination of these vectors equals the zero vector, i.e., if $c_1 v_1 + \dots + c_n v_n = \vec{0} \Rightarrow c_1 = \dots = c_n = 0$.
Otherwise they are linearly dependent (LD)

Note: • if $v_i = v_j$ for some $i \neq j \Rightarrow$ LD
- if $v_i = \vec{0}$ for some $i \Rightarrow$ LD
- Gaussian elimination - zero row in REF $\Rightarrow$
  $\Rightarrow$ original rows are LD.