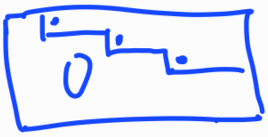
Review: • $Ax=b$ • EROs … $t\otimes_i$. $\oplus_i^j$. $\circlearrowleft_j^i$ | LA1 16/10/2025

• EROs do not change the solution set.



• REF, pivots, pivot/free columns
pivot/free variables

• Row reduction algorithm $(A|b) \to \text{REF } (A'|b')$

THM: 1: pivot in b' … no solution

otherwise 2a: pivot in every column of A' … $\exists!1$ solution

otherwise 2b: a free column in A': for any values for the free variables, there is a <u>unique</u> choice of values for pivot variables that produces a solution

Ex1.

$$(A'|b') = \begin{array}{c} \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 1 & 3 \\ & 1 & 2 & 1 & 0 & 1 \\ & & & 3 & 3 & 3 \end{array}\right) \\ \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \end{array}$$

• pivot columns/variable
• free column/variable

3rd equation: $3x_4+3x_5=3 \Rightarrow \underline{x_4} = 1 - x_5$

2nd equation: $x_2+2x_3+x_4 = 1$,
i.e. $\underline{x_2} = 1 - x_4 - 2x_3 = 0 - 2x_3 + x_5$

1st equation: $x_1+x_2+x_3+x_5 = 3$
i.e. $\underline{x_1} = 3 - x_2 - x_3 - x_5 = 3 + x_3 - 2x_5$

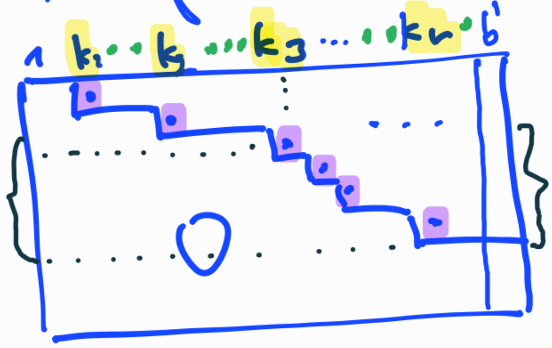
Proof of part 2b of THM:

b) let r denote the number of pivots of A'.

We will show <u>by induction</u>:

For each $i=r, r-1, \dots, 1$:

For <u>any values of free varibles x_j</u> with index <u>$j>k_i$</u>, the values of the pivot variables <u>$x_{k_i}, \dots, x_{k_r}$</u> are <u>uniquely determined</u> by equations $i, i+1, \dots, r$.



- base case: $i=r$

r-th equation:

$$\underbrace{\underset{\text{pivot } \neq 0}{a'_{rk_r}}\cdot x_{k_r} + \sum_{j=k_r+1}^{n} a'_{rj}\cdot \overset{\text{free variables only}}{x_j}}_{} = b'_r$$

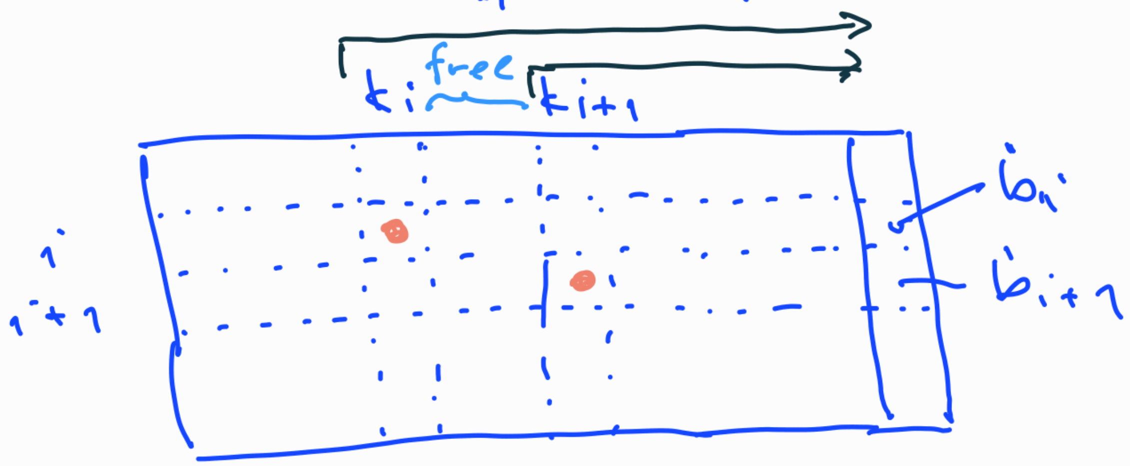
for any choice of free variables x_j, $j>k_r$, x_{k_r} is uniq. determined.

- induction step: $i+1 \Rightarrow i$ (backwards!)

i-th equation:

$$\underbrace{\underset{\text{pivot } \neq 0}{a'_{ik_i}}\cdot x_{k_i} + \overbrace{\sum_{j=k_i+1}^{k_{i+1}-1} a'_{ij}\cdot x_j}^{\text{free var. only}} + \overbrace{\sum_{j=k_{i+1}}^{n} a'_{ij}\cdot x_j}^{\text{by ind. assumption, all pivot variables uniquely determined}} = b'_i}_{}$$

$\Rightarrow$ pivot variable x_{k_i} uniquely determined as well



<u>Ex1</u> - continuation. An alternative description of the solution set. Let p_1 be the value free v. x_3,
p_2 — " — x_5

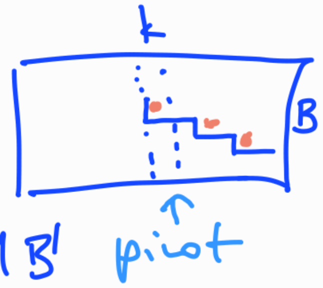
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + p_1 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + p_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

2 parameters
"
n − # of pivots = 5−3

Back substitution – the method of solving $A'x=b'$ in row echelon form by starting from the last variable, then substituting it in the above equations, etc.; as in the previous proof.

Corollary (uniqueness of pivot positions):
For any two matrices B and B' in REF obtained by elementary row operations from A, the pivot positions are the same.

Proof: By contradiction. Assume the claim does not hold for A, B, B' $\rightsquigarrow$ the pivot columns are different in B, B'.
Let k be the largest index s.t.



- $\forall j > k$, the character (pivot/free) of the j-th column is the same in B and B'

- k-th column in B is pivot and is free in B' (wlog)

Consider systems of lin. eq. $Ax=0$, $Bx=0$, $B'x=0$.
By Lemma 1, they have the same set of solutions.
However
- for $Bx=0$, the value of x_k is uniquely determined by the choice of free variables x_j, $j>k$,
- for $B'x=0$, any value can be used for x_k (by previous Theorem) – a contradiction. ∎

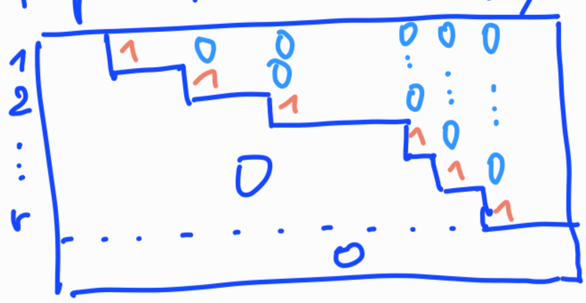
Def: The rank of a matrix A, denoted by $\operatorname{rank}(A)$, is the number of pivots in any matrix in REF obtained from A by elem. row operations.

Note: a meaningfull definition, due to Corollary.

Def: Reduced row echelon form of a matrix (RREF).

special form of REF: i) all pivots are $=1$,

ii) above every pivot are all 0's



HW: How to obtain RREF from REF by ERO's.

"Gauss-Jordan elimination"

Def: Homogeneous system of linear equations:

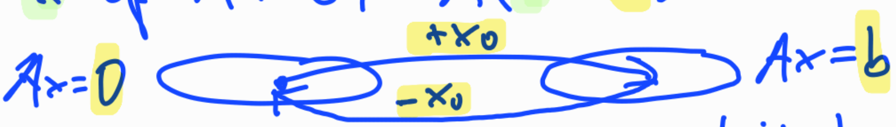
$$Ax=0 \qquad \left(\text{i.e., } b=\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}\right)$$

Observation: Let x_0 be a fixed solution of $Ax=b$.

Then • For every solution x' of $Ax=0$, $A(x_0+x')=b$,

• for every solution $\bar{x}$ of $Ax=b$, $A(\bar{x}-x_0)=0$.

Proof - HW.



Meaning: the mapping $g(x)=x+x_0$ is a bijection between solution sets of $Ax=0$ and $Ax=b$.

(injective + surjective = bijective)

Theorem (Structure of the solution set).

Let S be the solution set of $Ax=b$ with n variables, m equations, and let $r=\operatorname{rank}(A)$. If $S\neq\emptyset$, then there exist solutions $u_1,\dots,u_{n-r}$ of $Ax=0$ and a solution z of $Ax=b$, s.t. $S=\{z+p_1u_1+\dots+p_{n-r}\cdot u_{n-r} : p_1,\dots,p_{n-r}\in\mathbb{R}\}$.

COMPUTING WITH MATRICES

Df: A **real $m \times n$ matrix** is an ordered $(m \cdot n)$-tuple of real numbers arranged in rows in columns:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

i-th row : $a_{i1} \cdots a_{in}$ $\quad A_{i*}$

j-th column : $\begin{matrix} a_{1j} \\ \vdots \\ a_{mj} \end{matrix}$ $\quad A_{*j}$

an element in row i, column j : a_{ij} or A_{ij}

Df: A real **n-dimensional column vector** is $n \times 1$ matrix.
———— " ———— **row vector** is $1 \times n$ matrix.

zero $(m \times n)$ matrix ... **0** - all entries are 0

Matrix addition : the sum **$A+B$** of $m \times n$ matrices A, B is $m \times n$ matrix C s.t. $C_{ij} = A_{ij} + B_{ij}$, $\forall i, j$

Scalar multiplication of an $m \times n$ matrix A:
For $k \in \mathbb{R}$, **$k \cdot A$** is an $m \times n$ matrix D s.t. $D_{ij} = k \cdot A_{ij}$ $\forall i, j$

Equality of matrices : **$A = B$**, for $m \times n$ matrices, if $A_{ij} = B_{ij}$, $\forall i, j$

Lemma:

For matrices $A, B, C \in \mathbb{R}^{m \times n}$ and scalars $s, t \in \mathbb{R}$, the following holds:

1. $A + B = B + A$ — **commutative** property
2. $(A+B) + C = A + (B+C)$ — **associative** property
3. $A + 0 = A$
4. $A + (-A) = 0$, where $-A = (-1) \cdot A$
5. $s \cdot (t \cdot A) = (s \cdot t) \cdot A$
6. $s(A+B) = s \cdot A + s \cdot B$
7. $(s+t) \cdot A = s \cdot A + t \cdot A$

$\left.\begin{matrix} 6. \\ 7. \end{matrix}\right\}$ **distributive** property

Proof: 1. We need to check that $\forall i,j$,

$$\begin{array}{ccc} (A+B)_{ij} & \overset{?}{=} & (B+A)_{ij} \\ \parallel & & \parallel \\ A_{ij}+B_{ij} & = & B_{ij}+A_{ij} \end{array}$$

← definition of matrix addition

↑ commutativity of addition of real numbers

2.–7. – analogously, HW.

∎