Review: $Ax=b$

LA1 9/10/2025

ELEMENTARY ROW OPERATIONS

a) scalar multiplication: multiply row i by a non-zero scalar t

b) add one row j to another row i $\quad \oplus_j^i$

b') row sum: add a multiple of one row to another row

c) row swap: exchange any two i, j rows $\quad \circlearrowleft_j^i$

Operations b') and c) can be realized by a combination of operations a) and b). HW!

E.g.
$$\left(\begin{array}{cccc|c} 0 & 3 & 3 & 3 & 3 \\ -1 & 1 & 0 & 2 & -2 \\ 1 & 0 & 1 & 2 & 0 \end{array}\right) \overset{c\ \circlearrowleft_2^1}{\rightsquigarrow} \left(\begin{array}{cccc|c} -1 & 1 & 0 & 2 & -2 \\ 0 & 3 & 3 & 3 & 3 \\ 1 & 0 & 1 & 2 & 0 \end{array}\right) \overset{b\ \oplus_1^3}{\rightsquigarrow} \left(\begin{array}{cccc|c} -1 & 1 & 0 & 2 & -2 \\ 0 & 3 & 3 & 3 & 3 \\ 0 & 1 & 1 & 4 & -2 \end{array}\right)$$

$$\overset{b'\ \oplus^3_{-\frac{1}{3}2}}{\rightsquigarrow} \begin{array}{c} \begin{array}{cccc} 1 & 2 & & 4 \end{array} \\ \left(\begin{array}{cccc|c} -1 & 1 & 0 & 2 & -2 \\ 0 & 3 & 3 & 3 & 3 \\ 0 & 0 & 0 & 3 & -3 \end{array}\right) \end{array} \qquad (r=3 \quad k_1=1 \quad k_2=2 \quad k_3=4)$$

Lemma 1: Let $(A'|b')$ be the result of applying an ERO to the augmented matrix $(A|b)$ (of $Ax=b$). Then a vector v satisfies $Av=b$ if and only if it satisfies $A'v=b'$.

informally: EROs do not change the solution set.

Proof: ERO of type a) - multiplication of row i

let S ... solutions of $Ax=b$, R ... solutions of $A'x=b'$

We want to show $S=R$.

Consider $v=(v_1,\dots,v_n)^T \in S$. Then v satisfies

- equations $1,\dots,i-1,i+1,\dots,m$ of $A'x=b'$ (no change)
- i-th new eq: $t\cdot a_{i1}v_1 + t\cdot a_{i2}v_2 + \dots + t\cdot a_{in}v_n \overset{?}{=} t\cdot b_i$,

as v satisfies: $a_{i1}v_1 + a_{i2}v_2 + \dots + a_{in}v_n = b_i$

$\Rightarrow S \subseteq R$

Note: $(A|b)$ can be obtained from $(A'|b')$ by ERO of type a), namely multiply i-th row by $\frac{1}{t} \neq 0$.

$\Rightarrow R \subseteq S$ $\qquad \Rightarrow R = S$.

For ERO of type b) - analogously. HW! ∎

GAUSSIAN ELIMINATION

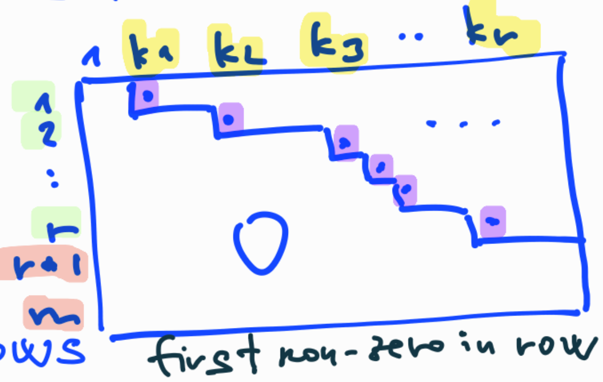
An algorithm for solving $Ax = b$.

Phase 1 transform $(A|b)$ by EROs into a special form called row echelon form

Phase 2 by back substitution we find all solutions

Def: Matrix A with m rows is in row echelon form (REF) if there exists $r \in \{0, 1, \dots, m\}$ s.t.

- rows $1, 2, \dots, r$ are non-zero,
- for $k_i := \min\{j \mid a_{ij} \neq 0\}$, $k_1 < k_2 < \dots < k_r$, and
- rows $r+1, \dots, m$ are all-zero rows



Pivots - the entries on positions $(1, k_1), (2, k_2) \dots (r, k_r)$

Pivot columns - columns with pivots, i.e., $k_1, k_2, \dots, k_r$

Free columns - all other (cf. the earlier example)

The corresponding variables: pivot (basic) / free

THE ROW REDUCTION ALGORITHM

1 Find the first non-zero column-index k_1.
If there is no such, STOP $(A \equiv 0)$.

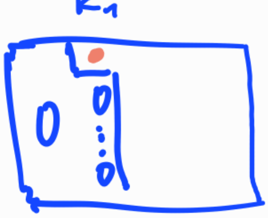
2 If $a_{1k_1} = 0$, swap row 1 with any row i s.t. $a_{ik_1} \neq 0$.

3 For $i = 2, 3, \dots, m$, add $\left(-\frac{a_{ik_1}}{a_{1k_1}}\right)$ multiple of row 1 to row i.

4 If $m \geq 2$, apply the same procedure on a submatrix without the first row; otherwise STOP.

(*) i) After step 2: $a_{1k_1} \neq 0$

ii) After step 3: $a_{2k_1} = a_{3k_1} = \dots = a_{mk_1} = 0$



iii) The algorithm will stop (as # of rows decreases).

Lemma: The algorithm transforms the matrix into REF.

Proof: By induction on the number of rows.

- base step: $m = 1$ - A has a single row - always in REF:

  either $r = 0$ and all entries are 0: $00\dots0$

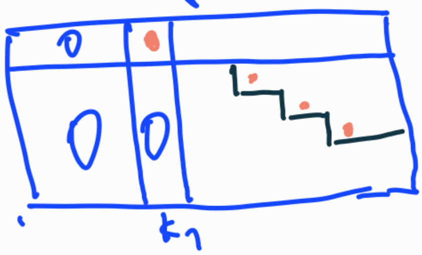
  or $r = 1$ and there is a non-zero somewhere

- induction step: $m-1 \to m$

  By (*) ii), the recursive call in step 4 will not affect columns $1, 2, \dots, k_1$.

  By inductive assumption, the recursive call transforms the submatrix consisting of lines $2, \dots, m$ to REF, and the first non-zero column is of index $> k_1$.

  $\Rightarrow$ The entire matrix is in REF.



Theorem (solution set of $Ax = b$).

Let $(A'|b')$ be a matrix in REF obtained by ERO from $(A|b)$.

1. If there is any pivot in the b' column, then the solution set of $Ax = b$ is empty.
2. Otherwise
   a) if A' has only pivot columns, then there is <u>exactly one</u> vector satisfying $Ax = b$.
   b) if there is a free column in A', then the solution set of $Ax = b$ is <u>infinite</u>.

   Moreover, given any values for the free variables, there is a <u>unique</u> choice of pivot variables that produces a solution.

Proof: 1. Assume b_i' is a pivot. Then the i-th equation of $A'x=b'$ is: $\underbrace{0\cdot x_1+\dots+0\,x_n}_{=0} = b_i' \neq 0$

There is no x satisfying this

$\Rightarrow$ no solution of $A'x=b' \overset{\text{lemma 1}}{\Rightarrow}$ no solution of $Ax=b$.

2. a) note: the pivot positions are $(1,1),(2,2),\dots,(n,n)$

We will show: for each $i=n,n-1,\dots,1$, the values of $x_i, x_{i+1},\dots,x_n$ of a solution x are uniquely determined by equations $i, i+1,\dots,n$.

by induction: • $i=n$ $\quad \underbrace{0x_1+0x_2+\dots+0x_{n-1}}_{=0} + \overset{\text{pivot}\neq 0}{a'_{nn}}x_n = b'_n$

$$\Rightarrow x_n = \frac{b_n}{a'_{nn}}, \text{ i.e., } x_n \text{ uniquely determined.}$$

• induction step: $i+1 \longrightarrow i$ (backwards!)

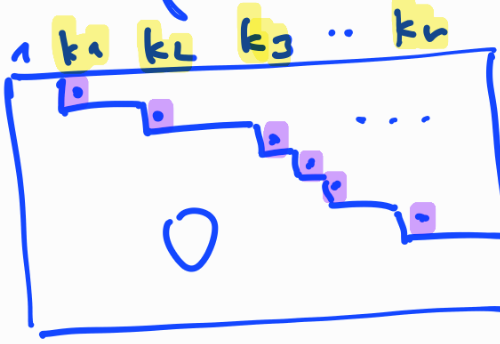
i-th eq: $\underbrace{0x_1+\dots+0x_{i-1}}_{=0} + \overset{\text{pivot}\neq 0}{a'_{ii}}x_i + \underbrace{a'_{i\,i+1}\overset{\text{uniquely determined - induct.}}{x_{i+1}}+\dots+a'_{in}x_n}_{=\Delta} = b'_i$

$$\Rightarrow x_i = \frac{b'_i-\Delta}{a'_{ii}}, \text{ i.e., } x_i \text{ uniquely determined.}$$

b) let r denote the number of pivots of A'.

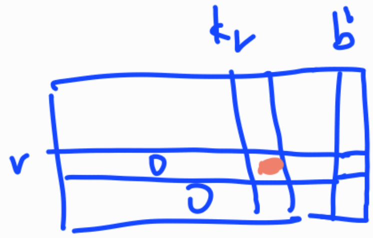
We will show: for each $i=r,r-1,\dots,1$:

for any values of free variables x_j with index $j>k_i$, the values of the pivot variables $x_{k_i},\dots,x_{k_r}$ are uniquely determined by equations $i,i+1,\dots,r$.



by induction: • $i=r$

$$\underbrace{\overset{\text{pivot}}{0\neq a'_{rk_r}}\cdot x_{k_r} + \sum_{j=k_r+1}^{n} a'_{rj}\cdot \overset{\text{free variables only}}{x_j} = b'_r}$$



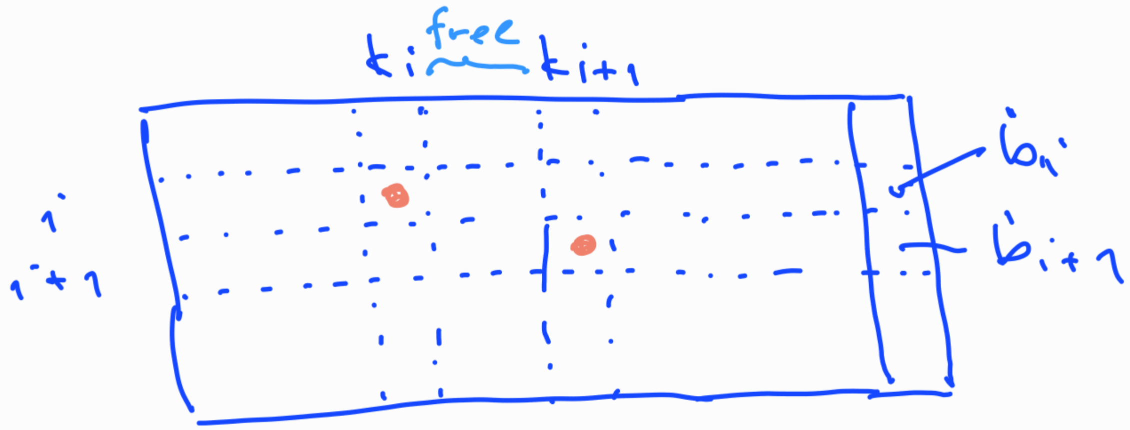
for any choice of free variables $x_j, j>k_r$, x_{k_r} is uniq. determined.

- induction step: $i+1 \to i$ (backwards!)

i-th equation:

$$\underbrace{\overset{\text{pivot}}{a'_{ik_i}} \cdot x_{k_i} + \overbrace{\sum_{j=k_i+1}^{k_{i+1}-1} a'_{ij}\cdot x_j}^{\text{free var. only}} + \overbrace{\sum_{j=k_{i+1}}^{n} a'_{ij}\cdot x_j}^{\text{by ind. assumption, all pivot variables determin}} = b'_i}, \quad a'_{ik_i} \neq 0$$

$\Rightarrow$ pivot variable x_{k_i} uniquely determined as well

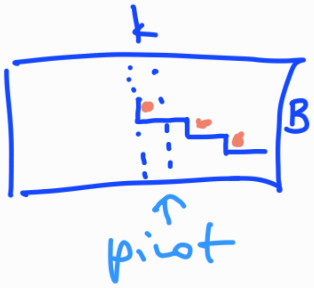


Back substitution – the method of solving $A'x=b'$ in row echelon form by starting from the last variable, then substituting it in the above equations, etc., as in the previous proof.

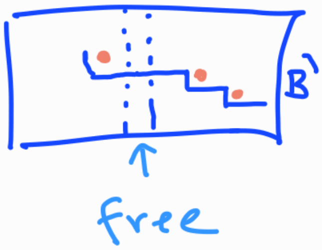
Corollary (uniqueness of pivot positions):
For any two matrices B and B' in REF obtained from A by elementary row operations, the pivot positions are the same.

Proof: By contradiction. Assume the claim does not hold for A, B, B'. Let k be the largest index s.t.

- $\forall j > k$, the character (pivot/free) of the j-th column is the same
- k-th column in B is pivot and is free in B' (wlog)



The systems $Ax=0$, $Bx=0$, $B'x=0$ have the same set of solutions, by Lemma 1.



However • for $Bx=0$, the value of x_k is uniquely determined by the choice of free variables $x_j, j>k$,
• for $B'x=0$, any value can be used for x_k (by previous Theorem) - a contradiction. ∎

Def: The rank of a matrix A, denoted by $\mathrm{rank}(A)$, is the number of pivots in any matrix in REF obtained from A by elem. row operations.

Note: a meaningfull definition, due to Corollary.