LINEAR TRANSFORMATIONS

Review A mapping $f : \underset{\text{domain}}{U} \to \underset{\text{host space}}{V}$ s.t. $\forall u, v \in U,\ \forall t \in T$: U, V .. VS over T

- $f(u+v) = f(u) + f(v)$
- $f(t u) = t \cdot f(u)$.

💡 $\forall n \in \mathbb{N}, \forall a_1, \dots, a_n \in T, \forall u_1, \dots, u_n \in U$: $f\left(\sum_{i=1}^{n} a_i u_i\right) = \sum_{i=1}^{n} a_i f(u_i)$

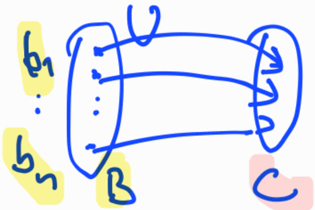
Def: isomorphism – bijective linear transformation
(bijective = surjective & injective)

THM1: Every VS V over T of dim n is isomorphic to T^n.

THM2: Images of a basis determine the linear transform.

Def: The matrix of LT f with respect to bases B and C:

$$ {}_C[f]_B = \left.\underbrace{\begin{pmatrix} | & & | \\ [f(b_1)]_C & \cdots & [f(b_n)]_C \\ | & & | \end{pmatrix}}_{|B| = n \text{ columns}}\right\} |C| \text{ rows} $$



THM3: $\forall u \in U$, $[f(u)]_C = {}_C[f]_B \cdot [u]_B$.

Def: Let B and C be two bases of a VS V of finite dimension. Then ${}_C[id]_B$ is called the change of basis matrix from B to C (or: transition matrix from B to C).

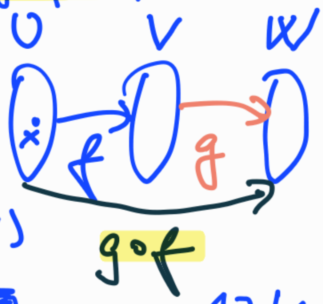
THM 4 (new): An isomorphism f (finite dim) maps basis to bases.

Proof outline
- we know that f maps a basis to SC.
- check that f maps LI vectors to LI vectors. H/W ☒

Corollary: If VS U and V are isomorphic, then $\dim(U) = \dim(V)$.

Def: For LTs $f : U \to V$ and $g : V \to W$, the composition of f and g is the mapping $g \circ f$ from U to W def. by $(g \circ f)(x) = g(f(x))$, $\forall x \in U$.

Lemma: $g \circ f$ is a linear transformation.



Proof: $\forall u, v \in U$: $(g \circ f)(u+v) \overset{\text{def.}}{=} g(f(u+v)) \overset{f \text{ is lin.}}{=}$

$g(f(u) + f(v)) \overset{g \text{ is lin.}}{=} g(f(u)) + g(f(v)) \overset{\text{def.}}{=} (g \circ f)(u) + (g \circ f)(v)$

Similarly for $(g \circ f)(t \cdot u)$. ☒

Theorem 5 (Matrix of composition of lin. transformations):
Let U, V and W be VS over $\mathbb{F}$ with bases B, C and D, and let $f: U \to V$ and $g: V \to W$ be linear transformations. Then

$$ {}_D[g\circ f]_B = {}_D[g]_C \cdot {}_C[f]_B . $$

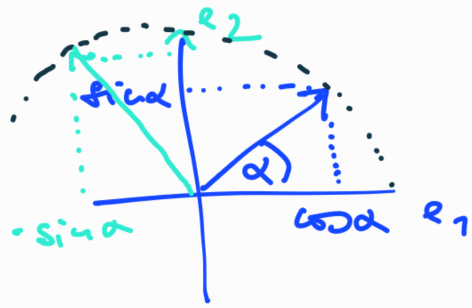
Proof: Consider a basis $B = (b_1, \dots, b_n)$ and the i-th column of the LHS:

$$\underbrace{[(g\circ f)(b_i)]_D}_{i\text{-th column of } {}_D[g\circ f]_B} \overset{\text{def. of compos.}}{=} [g(f(b_i))]_D \overset{\text{THM 3 for } g}{=} {}_D[g]_C \cdot [f(b_i)]_C \overset{\text{THM 3 for } f}{=} \underbrace{{}_D[g]_C \cdot {}_C[f]_B \cdot \overbrace{[b_i]_B}^{e_i}}_{i\text{-th column of the matrix product}}$$

∎

Example: Consider rotation by α, $f_\alpha: \mathbb{R}^2 \to \mathbb{R}^2$.
First, we express f_α w.r.t. the canonical basis $K = (e_1, e_2)$.

$$f_\alpha(e_1) = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \qquad f_\alpha(e_2) = \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}$$



$$(*) \quad {}_K[f_\alpha]_K = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

Consider $f = f_\beta \circ f_\alpha$ (i.e., rotation by $\alpha+\beta$)

Then i) $${}_K[f]_K = \begin{pmatrix} \cos(\alpha+\beta) & \vdots \\ \sin(\alpha+\beta) & \vdots \end{pmatrix}$$

ii) by $(*)$ and the previous THM:

$${}_K[f]_K = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \cdot \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta & \vdots \\ \cos\alpha \sin\beta + \sin\alpha \cdot \cos\beta & \vdots \end{pmatrix}$$

$\cos(\alpha+\beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta$

$\sin(\alpha+\beta) = \cos\alpha \sin\beta + \sin\alpha \cdot \cos\beta$

i.e., we derived the sin and cos sum formulas!

In If f is isomorphism, then f^{-1} is isomorphism.

Proof: check that f^{-1} is a lin. transf. HW ☒

Theorem: Let U and V be VS of finite dimension over T with bases B and C. Then a lin. transformation $f: U \to V$ is **isomorphism** $\iff$ ${}_C[f]_B$ is **invertible**.

Proof: $\Rightarrow$ for $B = (b_1, \dots, b_n)$, $f(b_1), \dots, f(b_n)$ is basis of V, (by THM 4)

thus, $[f(b_1)]_C, \dots, [f(b_n)]_C$ is also a basis, i.e., lin. indep.,

thus, ${}_C[f]_B$ is invertible.

$\Leftarrow$ consider a lin. transf. $g: V \to U$ s.t. ${}_B[g]_C = ({}_C[f]_B)^{-1}$.

Then

i) ${}_B[g \circ f]_B \overset{\text{by THM5}}{=} {}_B[g]_C \cdot {}_C[f]_B \overset{\text{by def. of } g}{=} I$, i.e., f is injective

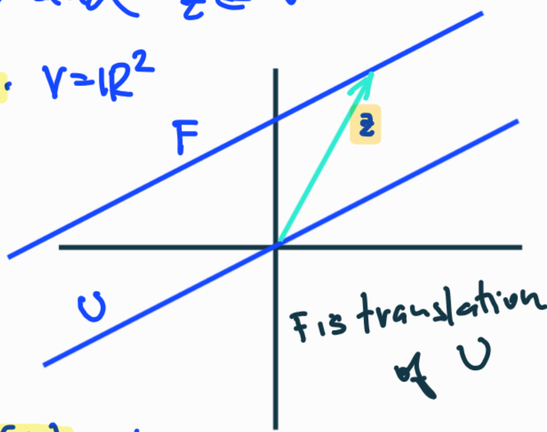
ii) ${}_C[f \circ g]_C = {}_C[f]_B \cdot {}_B[g]_C = I$, i.e., f is surjective

$\Rightarrow f$ is isomorphism ☒

Def: A subset F of a VS V over T is an **affine subspace** of V if F is of the form $F = \{z + u : u \in U\}$ where U is a vector subspace of V and $z \in V$.

We define $\dim(F) = \dim(U)$. E.g. $V = \mathbb{R}^2$

Notation: $F = z + U$



F is translation of U

Theorem: The solution set of $Ax = b$ is either empty, or it forms an affine subspace of the form $x_0 + N(A)$ where x_0 is any solution of $Ax = b$.

null space of A

Proof: THM about solutions - Lec. 3 & def. of affine subspace.

IMAGE COMPRESSION AND HAAR BASIS

$V = \mathbb{R}^4$, canonical basis: $K = \left(\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right)$

Haar basis: $H = \left(\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\-1\end{pmatrix} \right)$ can be generalized to higher dim.
(orthogonal)

Consider $_H[id]_K = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & -1/4 & -1/4 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \end{pmatrix} =: W$

For any $x \in \mathbb{R}^4$: $[x]_H = {}_H[id]_K \cdot [x]_K = W \cdot x =: u$

$u_1 = \frac{\sum x_i}{4}$ ... average of x_1, x_2, x_3, x_4

$u_2 = \frac{x_1+x_2}{4} - \frac{x_3+x_4}{4} = \frac{x_1+x_2}{2} - \frac{\sum x_i}{4}$ how much does average of x_1, x_2 differ from average of x_1, x_2, x_3, x_4. e.g., if all x_i are equal, $u_2 = 0$

etc. ...

Consider a small 4×4 grayscale image A with 256 shades (8bits) per pixel

Ex. $A = \begin{pmatrix} 20 & 22 & 24 & 26 \\ 18 & 20 & 22 & 24 \\ 16 & 18 & 20 & 22 \\ 14 & 16 & 18 & 20 \end{pmatrix}$

Goal: Compress A without significant quality loss

Idea: use Haar basis

1. Represent columns of A in basis H: $A' = W \cdot A$
2. Represent rows of A' in basis H: $A'' = (W \cdot A) \cdot W^T$
3. Change small values in A'' to zero ⇒ matrix $\bar{A}$ new representation

(4. Return to canonical basis, if needed: $\bar{\bar{A}} = W^{-1} \bar{A} (W^{-1})^T$)

Note: without step 3 - reversible process

Ex. contd. ... we get $A'' = \begin{pmatrix} 20 & -2 & -1 & 1 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ even without zeroing - most of the values are 0's!

Compression of real images:
separate them into 8×8 sub-matrices and apply this or similar algorithm to each of them