LINEAR TRANSFORMATIONS

LA1 18|12|25

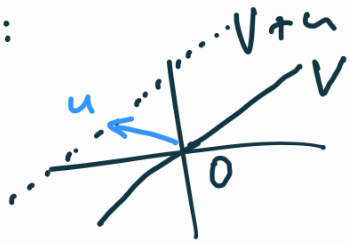
Review A mapping $f: U \to V$ s.t. $\forall u, v \in U$, $\forall t \in T$: U, V .. VS over T

- $f(u+v) = f(u) + f(v)$
- $f(tu) = t \cdot f(u)$.

Lemma 1: For a linear transformation $f: U \to V$, $f(\underset{\text{null vector in } U}{0}) = \underset{\text{in } V}{0}$.
(VS U and V are over field T)

Proof: $f(0) = f(\underset{\text{zero in } T}{0} \cdot \underset{\text{null vector in } U}{0}) \overset{\text{linearity}}{=} 0 \cdot f(0) = 0$ ← properties of VS ∎

Non-example: For VS V and $u \in V$, $u \neq 0$:
$f(x) = x + u$, $\forall x \in V$ - translation
- **not** a lin. transf.! $f(0) = u \neq 0$

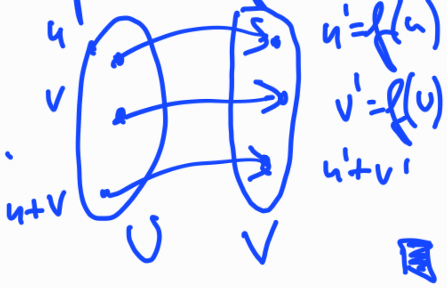


Notation: For LT $f: U \to V$, $f(U) = \{ f(u) : u \in U \}$.

Lemma 2: For a linear transformation $f: U \to V$, $f(U)$ is a subspace of V.

Proof: By lemma 1, $f(U) \neq \emptyset$ ⇒ it suffice to check that $f(U)$ is closed for $+$, $\cdot$.

$+$: for $u', v' \in f(U)$, there are $u, v \in U$ s.t. $f(u) = u'$, $f(v) = v'$.
⇒ by definition of lin. tran.:
$$u' + v' = f(u) + f(v) = f(u+v) \in f(U).$$



$\cdot$: analogously ∎

Definition: A <u>bijective</u> linear transformation is called an **isomorphism**. (one-to-one [injective] and onto [surjective])
"relabeling" of elements

Theorem: Every VS V over T of dimension n is isomorphic to the arithmetic VS T^n.

Proof: Consider *any* basis B of V, and the linear transformation $f(u) = [u]_B$ ... coordinates w.r.t. B

- injective (unique LC of B)
- surjective

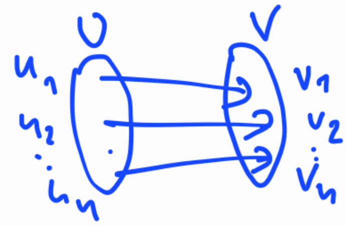
■

Observation: For vector spaces U and V over T, and for a linear transformation $f: U \to V$:

$$\forall n \in \mathbb{N}, \forall a_1, \dots, a_n \in T, \forall u_1, \dots, u_n \in U: f\left(\sum_{i=1}^{n} a_i u_i\right) \overset{\text{linear}}{=} \sum_{i=1}^{n} a_i f(u_i)$$

Proof: HW - by induction on n.

Theorem (Images of a basis determine the linear transform.)
Let U and V be VS over T, and $(u_1, \dots, u_n)$ be a basis of U. Then for every n-tuple $v_1, \dots, v_n$ there is exactly one LT $f: U \to V$ s.t. $\forall i: f(u_i) = v_i$.



Proof: Consider any $u \in U$. There is exactly one linear combination $u = \sum_{i=1}^{n} a_i u_i$

$$\Rightarrow f(u) = f\left(\sum_{i=1}^{n} a_i u_i\right) \overset{\text{Obser.}}{=} \sum_{i=1}^{n} a_i f(u_i)$$ i.e., $f(u)$ is specified ■

Corollary: For every LT $f: U \to V$, $\dim(f(U)) \le \dim(U)$.

Proof: For $f(U)$, there is a spanning collection of size $\dim(U)$ $\Rightarrow$ by Steinitz: $\dim(f(U)) \le \dim(U)$. ■

Theorem: Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then $f(x) = Ax$ where A is $m \times n$ matrix whose i-th column is $f(e_i)$ (where $e_i = (0, \dots, \overset{i\text{-th}}{1}, 0, \dots, 0)^T$).

Proof: Consider the canonical basis $e_1, \dots, e_n$ of VS $\mathbb{R}^n$. By the previous THM, using the definition of A:

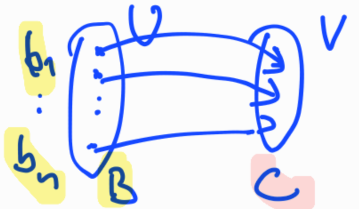
$$\forall x \in \mathbb{R}^n \quad f(x) = f\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} x_i f(e_i) = A \cdot x.$$ ■

Meaning: Linear transformation $\sim$ matrix multiplication. 12/2

Recall the $f: \mathbb{R}^2 \to \mathbb{R}^2$ examples from last week.

Def: Let U and V be VS over T with bases $B = (b_1, \dots, b_n)$ and $C = (c_1, \dots, c_m)$, and $f: U \to V$ be a linear transformation.
The matrix of f with respect to bases B and C is an $m \times n$ matrix

$$ {}_C[f]_B = \begin{pmatrix} | & & | \\ [f(b_1)]_C & \cdots & [f(b_n)]_C \\ | & & | \end{pmatrix}. $$



(columns of A: coordinates of images of B w.r.t. C)

Theorem: For a linear transformation $f: U \to V$ between VS U and V with bases $B = (b_1, \dots, b_n)$ and C,
it holds: $\quad \forall u \in U \quad [f(u)]_C = {}_C[f]_B \cdot [u]_B$.

Proof: Let us denote $[u]_B = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. Then

$$ [f(u)]_C = \left[f\left(\sum_{i=1}^n x_i \cdot b_i \right) \right]_C \overset{f \text{ is LT}}{=} \left[\sum_{i=1}^n x_i \cdot f(b_i) \right]_C \overset{\text{"Coordinates" is LT}}{=} $$

$$ \sum_{i=1}^n x_i \cdot [f(b_i)]_C = {}_C[f]_B \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = {}_C[f]_B \cdot [u]_B. \quad \blacksquare $$

$$ \begin{tikzcd} U \ni u \arrow[r, "f"] \arrow[d] & f(u) \in V \arrow[d] \\ T^n \ni [u]_B \arrow[r, "{}_C[f]_B"] & [f(u)]_C \in T^m \end{tikzcd} $$

Def: Let B and C be two bases of a VS V of finite dimension.
Then ${}_C[\mathrm{id}]_B$ is called the change of basis matrix from B to C (or: transition matrix from B to C).

Example: Consider $f: \mathbb{R}^3 \to \mathbb{R}^2$ defined as

$$f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 + 3x_3 \\ 4x_1 \qquad\quad + x_3 \end{pmatrix}.$$

Consider bases $B = \Big(\underset{b_1}{\begin{pmatrix}1\\0\\0\end{pmatrix}}, \underset{b_2}{\begin{pmatrix}0\\1\\0\end{pmatrix}}, \underset{b_3}{\begin{pmatrix}0\\0\\1\end{pmatrix}}\Big)$, $B' = \Big(\underset{b_1'}{\begin{pmatrix}1\\2\\3\end{pmatrix}}, \underset{b_2'}{\begin{pmatrix}0\\1\\2\end{pmatrix}}, \underset{b_3'}{\begin{pmatrix}0\\0\\1\end{pmatrix}}\Big)$ VP $\mathbb{R}^3$,

canonical

$C = \Big(\underset{c_1}{\begin{pmatrix}1\\0\end{pmatrix}}, \underset{c_2}{\begin{pmatrix}0\\1\end{pmatrix}}\Big)$, $C' = \Big(\underset{c_1'}{\begin{pmatrix}2\\1\end{pmatrix}}, \underset{c_2'}{\begin{pmatrix}1\\0\end{pmatrix}}\Big)$ VP $\mathbb{R}^2$.

Then:

$${}_C[f]_B = \begin{pmatrix} | & | & | \\ [f(b_1)]_C & [f(b_2)]_C & [f(b_3)]_C \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \end{pmatrix} =: A$$

by THM $\Rightarrow$ $\underbrace{[f(u)]_C}_{=f(u)} = \underbrace{{}_C[f]_B}_{A} \cdot \underbrace{[u]_B}_{=u}$ i.e., $f(u) = A \cdot u$

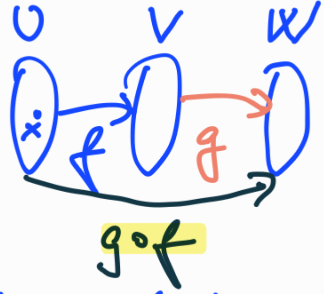
$f(b_1') = \begin{pmatrix}14\\7\end{pmatrix} = 7 \cdot c_1' + 0 \cdot c_2' \quad \Rightarrow \quad [f(b_1')]_{C'} = \begin{pmatrix}7\\0\end{pmatrix}$

$f(b_2') = \begin{pmatrix}8\\2\end{pmatrix} = 4 \cdot c_1' - 2 \cdot c_2' \quad \Rightarrow \quad [f(b_2')]_{C'} = \begin{pmatrix}4\\-2\end{pmatrix}$

$f(b_3') = \begin{pmatrix}3\\1\end{pmatrix} = c_1' + c_2' \quad \Rightarrow \quad [f(b_3')]_{C'} = \begin{pmatrix}1\\1\end{pmatrix}$

$$\Rightarrow {}_{C'}[f]_{B'} = \begin{pmatrix} 7 & 4 & 1 \\ 0 & -2 & 1 \end{pmatrix}$$

Def: For a linear transformation $f: U \to V$ and a LT $g: V \to W$, the composition of f and g is the mapping $g \circ f$ from U to W defined by $(g \circ f)(x) = g(f(x))$, $\forall x \in U$.



Lemma: $g \circ f$ is a linear transformation. HW!

Proof: For every $u, v \in U$:

$$(g \circ f)(u+v) \overset{\text{def.}}{=} g(f(u+v)) \overset{f \text{ is lin.}}{=} g(f(u) + f(v)) \overset{g \text{ is lin.}}{=} g(f(u)) + g(f(v))$$
$$\overset{\text{def.}}{=} (g \circ f)(u) + (g \circ f)(v)$$

Similarly for $(g \circ f)(t \cdot u)$.

Theorem (Matrix of composition of lin. transformations):
Let U, V and W be VS over T with bases B, C and D, and let $f: U \to V$ and $g: V \to W$ be linear transformations. Then

$$ {}_D[g \circ f]_B = {}_D[g]_C \cdot {}_C[f]_B . $$

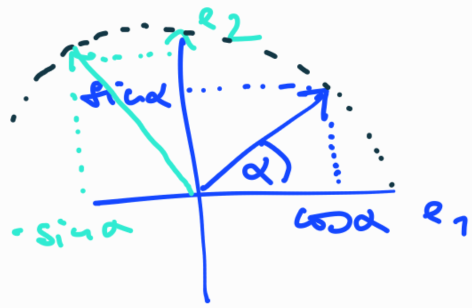
Proof: Consider the i-th vector b_i from the basis $B = (b_1, \dots, b_n)$.

$$ \underbrace{[(g\circ f)(b_i)]_D}_{i\text{-th column of } {}_D[g\circ f]_B} \overset{\text{def. compos.}}{=} [g(f(b_i))]_D \overset{\text{THM for } g}{=} {}_D[g]_C \cdot [f(b_i)]_C \overset{\text{THM for } f}{=} \underbrace{{}_D[g]_C \cdot {}_C[f]_B \cdot \overbrace{[b_i]_B}^{e_i}}_{i\text{-th column of the matrix product}} $$

∎

Example: Consider rotation by α, $f_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$.
First, we express f_α w.r.t. the canonical basis $K = (e_1, e_2)$.

$$ f_\alpha(e_1) = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \qquad f_\alpha(e_2) = \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix} $$



$$ (*) \quad {}_K[f_\alpha]_K = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} $$

Consider $f = f_\beta \circ f_\alpha$ (i.e., rotation by $\alpha + \beta$)

Then i) $${}_K[f]_K = \begin{pmatrix} \cos\alpha+\beta & \vdots \\ \sin\alpha+\beta & \vdots \end{pmatrix}$$

ii) by (*) and the previous THM:

$$ {}_K[f]_K = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \cdot \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = $$

$$ = \begin{pmatrix} \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta & \vdots \\ \cos\alpha \sin\beta + \sin\alpha \cdot \cos\beta & \vdots \end{pmatrix} $$

$\cos(\alpha+\beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta$

$\sin(\alpha+\beta) = \cos\alpha \sin\beta + \sin\alpha \cdot \cos\beta$

i.e., we derived the sin and cos sum formulas!