Theorem (Steinitz): If $I=(w_1,\dots,w_m)$ are LI and $S=(v_1,\dots,v_n)$ is a SC of a VS V, then $m\le n$, and the set I can be extended by $n-m$ vectors from S into a SC of V.

- Dimension of VS: cardinality of a basis
- Uniqueness of Linear Combinations w.r.t. a basis

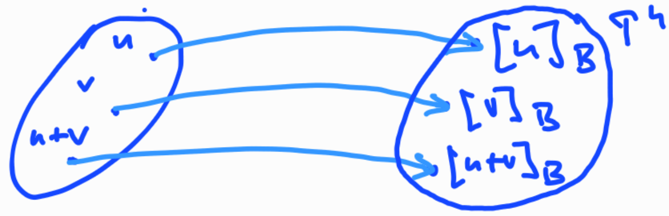
Def: The coordinates of $w\in V$ w.r.t. basis $B=(v_1,\dots,v_n)$ of a VS V, over T: $a_1,\dots,a_n\in T$ s.t. $w=\sum_{i=1}^n a_i v_i$. $[w]_B=\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}$

Lemma: Suppose B is a basis of a VS V over T, $n=\dim(V)$. Then $\forall\, u,v\in V,\ \forall \alpha\in T$: $[u+v]_B=[u]_B+[v]_B$

b. oper. in V — b. operation in T^n

$[\alpha\cdot v]_B=\alpha\cdot[v]_B$ HW

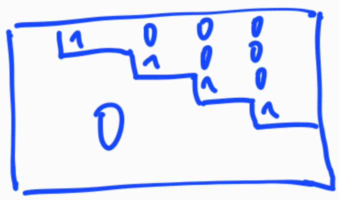
„Coordinates w.r.t. B" is a mapping $V\to T^n$ respecting the structure!

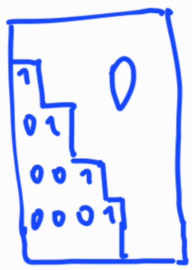


Subspaces associated with matrices

Theorem: For every matrix A, $\operatorname{rank}(A)=\operatorname{rank}(A^T)$.

Proof: i) Assume first that $A\in T^{m\times n}$ is in RREF

$A=$ 

$A^T=$ 

0, 1 – neutral elements in T w.r.t. $+,\cdot$

The rows with pivots are a basis of $R(A)$.

proof: LI, SC □ i.e., $\dim(R(A))=\operatorname{rank}(A)$

The columns with pivots are a basis of $C(A)$.

proof: LI, SC □

$\Rightarrow \operatorname{rank}(A)=\dim(R(A))=\dim(C(A))=\dim(R(A^T))=$
$=\operatorname{rank}(A^T)$.

Given an arbitrary matrix $A \in T^{m\times n}$, let R be an invertible matrix $T^{m\times m}$ s.t. $R\cdot A$ is in RREF.

Then

$$\text{rank}(A) = \text{rank}(R\cdot A) \overset{R\cdot A \text{ in RREF}}{=} \dim(C(R\cdot A)) \overset{\text{previous THM}}{=} \dim(C(A)) =$$

$$= \dim(R(A^T)) = \text{rank}(A^T) \qquad \blacksquare$$

Corollary: For every matrix A, $\dim(R(A)) = \dim(C(A))$.

Proof: Note that $R(A) = R(R\cdot A) \Rightarrow$
$\dim(R(A)) = \text{rank}(A)$; we know $\text{rank}(A) = \dim(C(A))$. $\blacksquare$

Theorem: For every $m\times n$ matrix A, $(\in T^{m\times n})$

$$\dim(\text{Null}(A)) + \text{rank}(A) = n.$$

Proof: Recall the theorem about the structure of solution set of $Ax=0$ - it is the linear span of properly chosen $n-\text{rank}(A)$ LI vectors - - i.e., $\dim(N(A)) = n - \text{rank}(A)$. $\square$

Alternative proof:

Consider any basis of $\text{Null}(A)$: $B = (v_1, \dots, v_k)$, $\forall i: v_i \in T^n$, and any extension of B to a basis of T^n: $v_{k+1}, \dots, v_n$ and a matrix R s.t. $R\cdot A$ is in REF (R is invertible!)

Goal: see that $Rv_{k+1}, \dots, Rv_n$ is a basis of $C(A)$:

LI? If $0 = \sum_{i=k+1}^{n} \alpha_i R v_i = R \cdot \sum_{i=k+1}^{n} \alpha_i v_i \rightarrow \sum_{i=k+1}^{n} \alpha_i v_i = 0 \rightarrow \forall i; \alpha_i = 0$ ✓

SC of $C(A)$? consider any $w \in C(A)$ w is LC of columns of A

$\Rightarrow w = Au$, for some $u \in T^n$.

We have a basis of $T^n \Rightarrow \exists \alpha_1, \dots, \alpha_n \in T$ s.t. $u = \sum_{i=1}^{n} \alpha_i v_i$

$$\Rightarrow w = A\cdot\sum_{i=1}^{n} \alpha_i v_i = \sum_{i=1}^{n} \alpha_i A v_i \underset{\uparrow}{=} \sum_{i=k+1}^{n} \alpha_i A v_i$$

$\Rightarrow Av_{k+1}, \dots, Av_n$ is SC ✓ $\qquad Av_i = 0$ for $i \leq n$! $\qquad \blacksquare$

Theorem: $\operatorname{rank}(A\cdot B) \le \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$, if $A\cdot B$ defined.

Proof: Note $R(AB) \subseteq R(B) \Rightarrow$

previous THM $\underline{\dim(R(AB))} \le \dim(R(B)) = \operatorname{rank}(B)$

Analogously $\quad \| \quad C(AB) \subseteq C(A) \Rightarrow$

$\operatorname{rank}(AB) = \underline{\dim(C(AB))} \le \dim(C(A)) = \operatorname{rank}(A)$

$\blacksquare$

Def: Let U and W be subspaces of VS V over T. Then the sum of U and W is the set $U+W = \{u+w : u\in U, w\in W\}$.

Lemma: If U and W are subspaces of V, then $U+W = \operatorname{span}(U\cup W)$.

Proof: $\subseteq$ obvious: for any $u\in U, w\in W$, $u+w \in \operatorname{span}(U\cup W)$.

$\supseteq$ consider arbitrary $z\in\operatorname{span}(U\cup W)$. By definition of span, z is a LC of finite number of vectors from $U\cup W$, say

$$z = \underbrace{\sum_{i=1}^{k}\alpha_i u_i}_{\in U} + \underbrace{\sum_{j=1}^{l}\beta_j w_j}_{\in W}, \text{ for } u_i\in U, w_j\in W, \alpha_i, \beta_j\in T, \forall i,j$$

$\Rightarrow z\in U+W$ $\blacksquare$

Theorem: Let U and W be subspaces of finite dimensions. Then $\dim(U+W) + \dim(U\cap W) = \dim(U) + \dim(W)$.

Proof: let $z_1,\dots,z_k$ be a basis of $U\cap W$ (it is a subspace)

By Steinitz theorem, there exists bases

$C = (z_1,\dots,z_k, v_1,\dots,v_l)$ of U

$D = (z_1,\dots,z_k, w_1,\dots,w_p)$ of W

Goal: see that $z_1,\dots,z_k, v_1,\dots,v_l, w_1,\dots,w_p$ is basis of $U+W$.

- LI: By contradiction. Assume $\exists$ nontrivial LC equal 0,

i.e. $\overbrace{\sum_{i=1}^{k}\alpha_i z_i + \sum_{i=1}^{l}\beta_i v_i}^{\in V} + \sum_{i=1}^{p}\gamma_i w_i = 0$ As C and D are LI, some $\beta_i \neq 0$, some $\gamma_i \neq 0$

$= \overbrace{-\sum_{i=1}^{p}\gamma_i w_i}^{\in W} \neq 0$

$\| \leftarrow \in V\cap W$

$\sum_{i=1}^{k}\alpha_i' z_i + \sum_{i=1}^{l} 0\cdot v_i$ - LC of vectors $z_1,\dots,z_k$ only

$\Rightarrow$ two different LC of basis vectors - a contradiction with uniqueness of LC

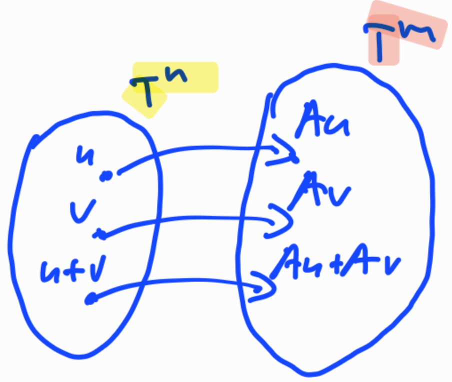
- SC - obvious

$\blacksquare$

LINEAR TRANSFORMATIONS

Think about matrix multiplication: For $A \in T^{m \times n}$, $u, v \in T^n$ and $t \in T$, it holds:

- $A(u+v) = Au + Av$
- $A(tu) = t(Au)$



i.e., multiplication by A is a mapping respecting structure

We noticed: coordinates $\underline{\qquad\qquad} \; \| \; \underline{\qquad\qquad}$.

Def: Let U and V be vector spaces over the same field T.
A mapping $f: U \to V$ is a **linear transformation** if
$\forall u, v \in U, \forall t \in T$:

- $f(u+v) = f(u) + f(v)$
- $f(tu) = t \cdot f(u)$.

Examples:

① For any VS V: $f(x) = 0 \quad \forall x \in V$ — **null transformation** — $f: V \to \{0\}$

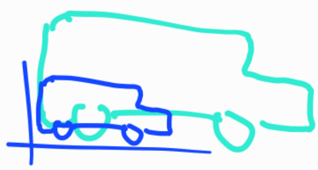
② For any VS V: $f(x) = x \quad \forall x \in V$ — **identity** — $f: V \to V$

③ For arithmetic VS $V = T^n$: $f\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_i, \; \forall \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in T^n$ — **projection** — $f: T^n \to T$

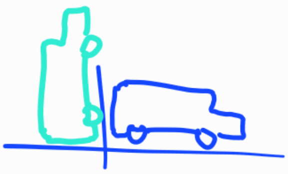
④ For any VS V of dimension n over T and any basis B:
$f(x) = [x]_B, \; \forall x \in V$ — **coordinates** — $f: V \to T^n$

⑤ For $V = \mathbb{R}^2$: $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$ — **reflection**

⑥ $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$ — **scaling**



⑦ $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$ — **rotation by $\frac{\pi}{2}$**



⋮

Lemma 1: For a linear transformation $f: U \to V$, $f(0) = 0$.
(VS U and V are over field T)
null vector in U ... in V

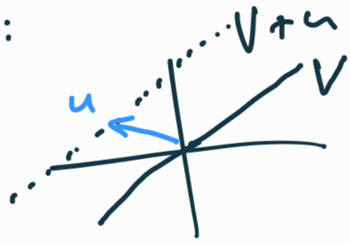
Proof: $f(0) = f(\underset{\text{zero in } T}{0} \cdot \underset{\text{null vector in } U}{0}) \overset{\text{linearity}}{=} 0 \cdot f(0) = 0$ ← properties of VS ∎

Lemma 2: For a linear transformation $f: U \to V$, $f(U)$ is a subspace of V.

Proof: By lemma 1, $f(U) \neq \emptyset$ ⇒ it suffice to check that $f(U)$ is closed for $+, \cdot$.

$+$: for $u', v' \in f(U)$, there are $u, v \in U$ s.t. $f(u) = u'$, $f(v) = v'$.
⇒ by definition of lin. tran.:
$$u' + v' = f(u) + f(v) = f(u+v) \in f(U).$$
$\cdot$: analogously ∎

Non-example: For VS V and $u \in V$, $u \neq 0$:
$$f(x) = x + u, \quad \forall x \in V.$$
translation – not a lin. transf.!



Definition: A bijective linear transformation is called an **isomorphism**. (one-to-one [injective] and onto [surjective])

Theorem: Every VS V over T of dimension n is isomorphic to the arithmetic VS T^n.

Proof: Consider any basis B of V, and the linear transformation $f(u) = [u]_B$ ... coordinates w.r.t. B
- injective
- surjective

∎