Review

Def: A collection of vectors $v_1, \ldots, v_n$ is linearly independent (LI) if $a_1v_1 + \cdots + a_nv_n = \vec{0} \Rightarrow a_1 = \ldots = a_n = 0$ $(\in T)$.

Note: an empty set $\emptyset$ of vectors is LI: (no coefficients) only empty sum $\Rightarrow$ in it all coefficients are 0.

Def: A set B of vectors is a basis of VS V if
i) B is LI, and ii) $\mathrm{span}(B) = V$.

Lemma (exchange): Let $B = \{v_1, \ldots, v_n\}$ be a spanning collection (SC) of VS V over T, and $w = \sum_{i=1}^{n} a_iv_i$ for some $a_1, \ldots, a_n \in T$. Then for each i s.t. $a_i \neq 0$, $B \setminus \{v_i\} \cup \{w\}$ is also SC of V.

Theorem (Steinitz exchange lemma): If $I = (w_1, \ldots, w_m)$ are linearly independent vectors and $S = (v_1, \ldots, v_n)$ is a spanning collection of a VS V, then $m \leq n$, and the set I can be extended by $n-m$ vectors from S into a spanning collection of V.

("Any spanning collection is at least as large as any linearly independent set.")

Corollary: Every two bases of a finitely generated VS V have the same size.

Proof of Cor: Consider bases $B = (b_1, \ldots, b_m)$, $C = (c_1, \ldots, c_n)$.

As i) B is LI and C is SC $\Rightarrow m \leq n$
ii) C is LI and B is SC $\Rightarrow n \leq m$
$\Big\}\ m = n$ ∎

Def: The dimension of a finitely generated VS V is the size of any of its bases. Notation: $\dim(V)$.

Proof of theorem: By induction on m.

- $m=0$: $m \leq n$ ✓

by extending $I=\emptyset$ by all vectors from S, we get SC ✓

- inductive step: $m-1 \longrightarrow m$

by inductive assumption, $m-1$ vectors $w_1, \dots, w_{m-1}$ can be extended by $n-m+1$ vectors from S into a SC of V - say $S' = (w_1, \dots, w_{m-1}, \overbrace{v_{i_1}, \dots, v_{i_{n-m+1}}}^{*})$.

$\Rightarrow$ w_m can be expressed as a LC of vectors S'

(!) For at least one index $j \in \{1, \dots, n-m+1\}$, the coefficient of v_{i_j} in the LC is non-zero.

proof: Otherwise $w_1, \dots, w_m$ are LD - but they are LI. ∎

$\Rightarrow$ by the Exchange lemma, $S' \setminus \{v_{i_j}\} \cup \{w_m\}$ is SC.

Note: $m \leq |S'| = |S| = n$. ∎

Corollary 2: If W is subspace of finitely generated VS V, then $\dim(W) \leq \dim(V)$. If equality holds, then $V=W$.

Proof: consider a basis B of W, and a basis C of V.

Then: B … LI in V, C … SC, by Steinitz $\Rightarrow$ $\underset{\dim(W)}{|B|} \leq \underset{\dim(V)}{|C|}$

If $\dim(W) = \dim(V)$, then B can be extended by zero vectors from C to get a basis of V, i.e., B is a basis of V: $W = \operatorname{span}(B) = V$ ∎

Theorem (Uniqueness of Linear Combinations): Suppose $B = (v_1, \dots, v_n)$ is a **basis** of a VS V over T. Then every vector $w \in V$ may be **uniquely** written as a **LC** of vectors from B.

Proof. As $\operatorname{span}(B) = V$, $\forall w \in V$ is a LC of vectors from B.

- uniqueness: by contradiction: assume that

$\exists w \in V$ and $(\alpha_1, \dots, \alpha_n) \neq (\beta_1, \dots, \beta_n)$ s.t.

$$w = \sum_{i=1}^{n} \alpha_i w_i = \sum_{i=1}^{n} \beta_i w_i$$

Then $0 = w - w = \sum_{i=1}^{n} (\alpha_i - \beta_i) w_i$, i.e.,

zero vector as a non-trivial LC of B - a contradiction ∎

Def (Coordinates): The <u>coordinates</u> of a vector $w \in V$, <u>with respect to a basis</u> $B = (v_1, \dots, v_n)$ of a VS V, over T, are the coefficients $a_1, \dots, a_n \in T$ s.t. $w = \sum_{i=1}^{n} a_i v_i$.

Notation: $[w]_B = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

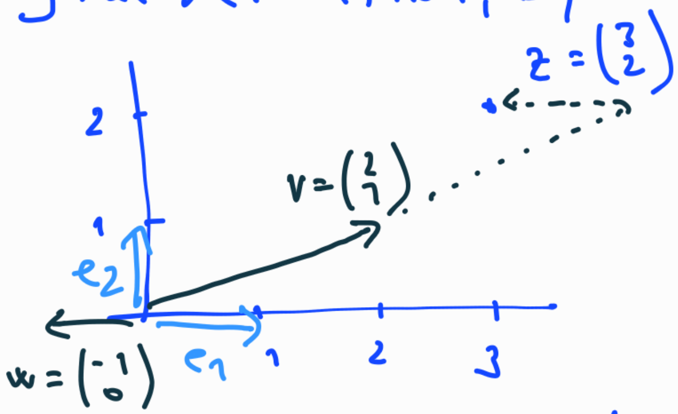
Note: a meaningful definition, by the previous THM.

Example: $V = \mathbb{R}^2$

$B = (e_1, e_2)$

$z = \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 3 \cdot e_1 + 2 \cdot e_2$

$[z]_B = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$



$B' = (v, w)$

$z = \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 2v + 1 \cdot w$

$[z]_{B'} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

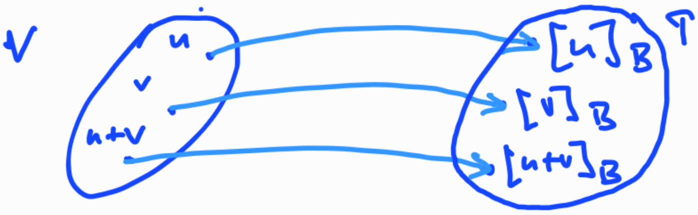
Note: Suppose B is a basis of a VS V over T, $n = \dim(V)$.

$\forall u, v \in V, \forall \alpha \in T$: $[u + v]_B = [u]_B + [v]_B$

$[\alpha \cdot v]_B = \alpha \cdot [v]_B$

b. oper. in V b. operation in T^n

„Coordinates w.r.t. B" is a mapping $V \to T^n$ respecting the structure!



SUBSPACES ASSOCIATED WITH MATRICES

We know: for a matrix A and invertible matrix R:

$$R(A) = R(R\cdot A) \qquad N(A) = N(R\cdot A) \qquad (*)$$

What about $C(A)$ and $C(R\cdot A)$?

Ex. $A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $R\cdot A = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow C(A) \neq C(R\cdot A)$

Theorem: For a matrix A and invertible R: $\dim(C(A)) = \dim(C(RA))$.

Lemma: If $(v_1, \dots, v_k)$ is a basis of $C(A)$, then $B' = (Rv_1, \dots, Rv_k)$ is a basis of $C(R\cdot A)$.

Note: the Lemma implies the Theorem - equal size bases.

Proof of Lemma: We check that i) $\mathrm{span}(B') = C(RA)$, and that B' are LI.

i) Let $u_1, \dots, u_n$ denote the columns of A,
$u_1', \dots, u_n'$ the columns of RA, i.e. $\forall i$, $u_i' = R\cdot u_i$

Consider $w \in C(RA)$. Then $w = \sum_{i=1}^n a_i \cdot u_i' = \sum_{i=1}^n a_i \cdot R \cdot u_i =$

$$= R \cdot \underbrace{\sum_{i=1}^n a_i u_i}_{\in C(A)} \underset{v_1,\dots,v_k \text{ is basis of } C(A)}{\overset{\text{for some } b_i\text{'s}}{=}} R\cdot \sum_{i=1}^k b_i v_i = \sum_{i=1}^k b_i (Rv_i) \quad \checkmark$$

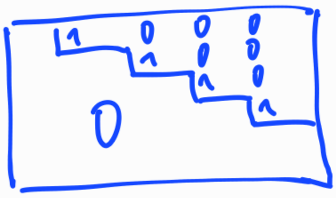
ii) by contradiction: assume that $Rv_1, \dots, Rv_k$ are LD, i.e., $\exists\, a_1, \dots, a_k$, not all zero, s.t. $\sum_{i=1}^k a_i R v_i = 0$

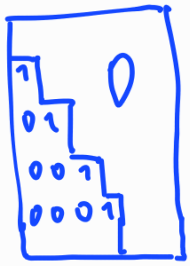
$$\Rightarrow R\cdot \sum_{i=1}^k a_i v_i = 0 \xRightarrow[\text{only trivial solution of } Rx=0]{R \text{ invertible}} \sum_{i=1}^k a_i v_i = 0 \text{ - a contradiction -}$$
- v_i's are LI. $\blacksquare$

For a matrix A: $\mathrm{rank}(A) = \dim(R(A))$ by $(*)$
$\mathrm{rank}(A^T) = \dim(C(A))$.

Theorem: For every matrix A, $\mathrm{rank}(A) = \mathrm{rank}(A^T)$.

Proof: i) Assume first that $A \in T^{m \times n}$, is in **RREF**

$A =$ 

$A^T =$ 

0, 1 – neutral elements in T w.r.t. $+, \cdot$

The rows with pivots are a basis of $R(A)$.

proof: LI, SC □

The columns with pivots are a basis of $C(A)$.

proof: LI, SC □

$\Rightarrow$ $\text{rank}(A) = \dim(R(A)) = \dim(C(A))$.

Given an arbitrary matrix $A \in T^{m \times n}$, let R be an invertible matrix $T^{m \times m}$ s.t. $R \cdot A$ is in RREF.

Then

$$\underline{\text{rank}(A)} = \text{rank}(R \cdot A) \overset{\text{part i)}}{=} \dim(C(RA)) \overset{\text{previous THM}}{=} \underline{\dim}(C(A)) =$$

$$= \dim(R(A^T)) = \underline{\text{rank}(A^T)}$$ ■

Theorem: For every $m \times n$ matrix A:

$$\dim(\text{Null}(A)) + \text{rank}(A) = n.$$