LINEAR ALGEBRA I - INTRO

$\boxed{\text{LA1 2/10/25}}$

Math (LA) - beauty & usefulness

LA useful for

- computer graphics, computer vision
- cryptography, coding (The beauty will show.)
- optimization
- data mining
- AI (machine learning)
- bioinformatics

(Intro review course ?!)

On teaching & learning

- not just WHAT? and HOW? questions,
  but mainly WHY? and WHY NOT ?
- abstraction - dealing with letters, not numbers
- intuition & exactness - both are needed
- our roles: I - to explain, you - to learn
- means: - hand written notes
  - write things down, do the math
  - tutorials
  - office hours - tutors, I
  - the course Math skills
  - student guides
  - mentoring sessions
- keep in mind: your value is not based
  on your academic performance
  (but do your best!)

UNDEMOCRATIC ELECTIONS

tennis club votes a new president

Rules: each member has a fractional vote

E.g. four members ($n=4$): Alice, Bob, Claire, David

Alice's vote:

$$v_A = \begin{pmatrix} 0 \\ 0 \\ 1/2 \\ 1/2 \end{pmatrix} \qquad v_B = \begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{pmatrix} \qquad v_C = \begin{pmatrix} 0 \\ 1/2 \\ 0 \\ 1/2 \end{pmatrix} \qquad v_D = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

In a table form:

colums - how people vote

row - how many unweighted votes people get

$$\begin{array}{c c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{pmatrix} 0 & 1/2 & 0 & 1 \\ 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 0 \\ 1/2 & 1/2 & 1/2 & 0 \end{pmatrix} \end{array} \begin{array}{c} \frac{3}{2} \\ \\ \\ \frac{3}{2} \end{array}$$

The outcome - n-tuple $H = (h_A, h_B, h_C, h_D) \neq (0,0,0,0)$
(indexed by the members)

such that for each member X, the dot product of H and the X's row of the table equals h_X.

e.g. $0 \cdot h_A + 1/2 \cdot h_B + 0 \cdot h_C + 1 \cdot h_d = h_A$ ... A

$0 \cdot h_A + 0 \cdot h_B + 1/2 \cdot h_C + 0 \cdot h_d = h_B$ ... B

The winner is the member X with the highest h_X.

e.g. $H = \left(\frac{8}{7}, \frac{2}{7}, \frac{4}{7}, 1\right) \Rightarrow$ Alice is the winner

Interpretation: h_X - the importance of the member X

- it depends on i) the importance of others, and ii) on the amount of votes they gave to X

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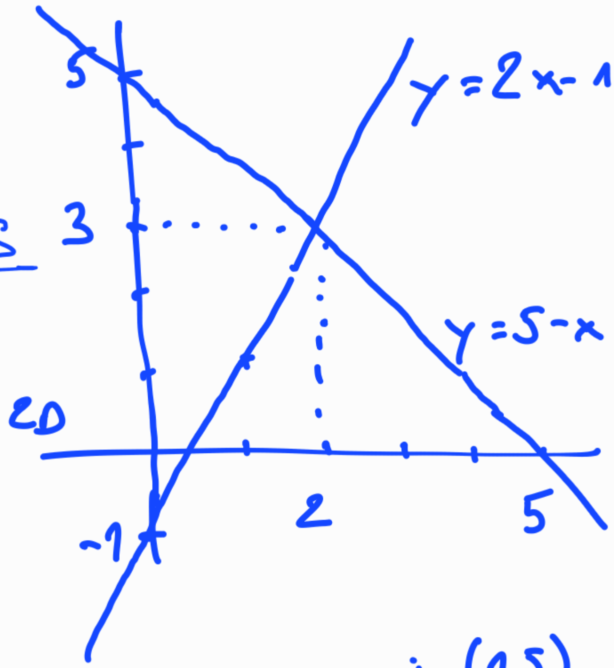
cf. PageRank - Google 1996

SYSTEMS OF LINEAR EQUATIONS

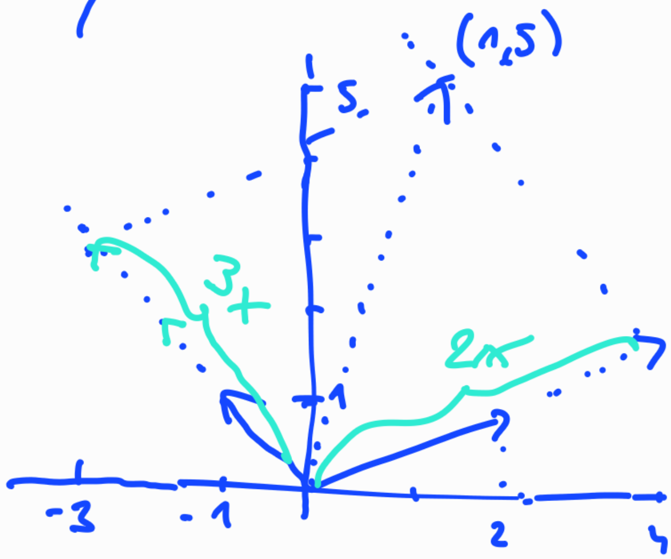
E.g. $2x - y = 1$
$x + y = 5$



- **one view**: focus on rows
  - equations ~ lines in 2D (planes in 3D, ...)

- **another view**: focus on columns

$$x\begin{pmatrix}2\\1\end{pmatrix} + y\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}1\\5\end{pmatrix}$$



  - we are looking for multiples of vectors $\begin{pmatrix}2\\1\end{pmatrix}$ and $\begin{pmatrix}-1\\1\end{pmatrix}$ that sum to $\begin{pmatrix}1\\5\end{pmatrix}$

Definition: A system of m linear equations with n variables is a system of m equations of the form:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2\\
&\vdots\\
a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m.
\end{aligned}$$

where a_{ij} are real (or rational, complex) numbers

- coefficients of the system $\quad i=1,\dots,m \quad j=1,\dots,n$

$b_1,\dots,b_m$ are real (or ...) numbers

- right-hand side

$x_1,\dots,x_n$ are the variables

Notation and terminology

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$

matrix of the system (i.e., an ordered $m \cdot n$ tuple)

a_{ij} ... the coefficient from row i, column j

A_{i-} ... i-th row (A_{i*}) $\quad A_{-j}$ ... j-th column (A_{*j})

$$b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$ vector of the right-hand side

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$ vector of variables

We write: $Ax = b$.

$$(A\,|\,b) = \left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & & & \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right)$$

augmented matrix of the system

Solution of the system $Ax=b$ is the set of all real (or rational, complex, ...) vectors $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ satisfying all m equations.

In our example: a single element set $\left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$.

ELEMENTARY ROW OPERATIONS (ERO)

a) scalar multiplication: multiply row i by a non-zero scalar t

b) add one row to another row

b') row sum: add a multiple of one row to another row

c) row swap: exchange any two rows

Operations b') and c) can be realized by a combination of operations a) and b). HW!

Lemma: Let $(A'|b')$ be the result of applying an ERO to the augmented matrix $(A|b)$ (of $Ax=b$). Then a vector v is solution of $Ax=b$ if and only if it is a solution of $A'x=b'$.

informally: EROs do not change the solutions.

Schematically:

original SLE	after ERO a	after ERO b
$A_{1-}x = b_1$	$\vdots$	$\vdots$
$\vdots$		
$A_{i-}x = b_i$	$t \cdot A_{i-}x = t \cdot b_i$	$A_{i-}x + A_{j-}x = b_i + b_j$
$\vdots$	$\vdots$	$\vdots$
$A_{j-}x = b_j$		
$\vdots$		
$A_{m-}x = b_m$		