LECTURE 8

19/11/2020

MAX SAT

RAND SAT	0,5 approximation	BEST PERFORMANCE FOR LONG CLAUSES
BIASED SAT	$\frac{\sqrt{5}-1}{2}$ approximation	
LP SAT	$\underbrace{(1-\frac{1}{e})}_{\sim 0{,}632}$ approximation	SHORT CLAUSES

Recall: clauses $C_1, C_2, \ldots, C_m$ variables $x_1, x_2, \ldots, x_n$

weights $w_1, w_2, \ldots, w_m$

ℓ_j ... lenght of C_j

for LP SAT: $\Pr[C_j \text{ satisfied}] \geq z_j^* \left(1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right)$

for RAND SAT: — " — $\geq (1 - 2^{-\ell_j})$

BEST SAT

with prob. $\frac{1}{2}$, run RAND SAT

with prob $\frac{1}{2}$, run LP SAT

Alternatively: run both, output the better solution

THM: The approximation ratio of BEST SAT is $\frac{3}{4}$.

Proof: W ... sum of weights of clauses satisfied by BEST SAT

W_1 LP SAT

W_2 RAND SAT

$$\mathbb{E}[W] = \mathbb{E}\left[\frac{1}{2}W_1 + \frac{1}{2}W_2\right] \geq$$

$$\geq \frac{1}{2}\sum_{j=1}^{m} w_j z_j^* \left(1-\left(1-\frac{1}{\ell_j}\right)^{\ell_j}\right) + \frac{1}{2}\sum_{j=1}^{m} w_j \left(1-2^{-\ell_j}\right)$$

$$\underset{z_j^* \leq 1}{\geq} \frac{1}{2}\sum_{j=1}^{m} w_j \cdot z_j^* \underbrace{\left[\left(1-\left(1-\frac{1}{\ell_j}\right)^{\ell_j}\right) + \left(1-2^{-\ell_j}\right)\right]}_{\geq \frac{3}{2}}$$

$$\begin{cases} \ell_j = 1 & 1 + \frac{1}{2} = \frac{3}{2} \\ \ell_j = 2 & (1-\frac{1}{4}) + (1-\frac{1}{4}) = \frac{3}{2} \\ \ell_j \geq 3 & \geq \underbrace{(1-\frac{1}{e})}_{\sim 0{,}632} + 0{,}875 \geq \frac{3}{2} \end{cases}$$

$$\geq \frac{3}{4} \sum_{j=1}^{m} w_j z_j^* \geq \frac{3}{4} OPT \quad \blacksquare$$

DERANDOMIZATION OF RAND SAT

Notation: for $\sigma \in \{t, f\}^n$ $\quad C_j(\sigma) = \begin{cases} 1 & \text{if } C_j \text{ satisfied by } \sigma \\ 0 & \text{NOT} \end{cases}$

$$W(\sigma) = \sum_{j=1}^{m} w_j \cdot C_j(\sigma)$$

W ... random variable ... sum of weights of clauses satisfied by RAND SAT

we known: $\frac{OPT}{2} \leq E[W]$

Given $C_1, C_2, \dots, C_m$, $w_1, \dots w_m$

Can we compute $E[W]$? YES!

$$= \frac{1}{2^n} \sum_{\sigma \in \{t,f\}^n} W(\sigma) = \sum_{j=1}^{m} w_j \cdot \underbrace{\boxed{\frac{1}{2^n} \sum_{\sigma \in \{t,f\}^n} C_j(\sigma)}}_{= Pr[C_j \text{ is satisfied by RAND SAT}]}$$

$$= \frac{1}{2^n} \left[\sum_{\sigma' \in \{t,f\}^{n-1}} \left[(W(f\sigma') + W(t\sigma') \right] \right]$$

$$= \frac{1}{2} \left[\underbrace{\frac{1}{2^{n-1}} \sum_{\sigma' \in \{t,f\}^{n-1}} W(f\sigma')}_{E[W \mid x_1 = f] = E[W \mid f]} + \underbrace{\frac{1}{2^{n-1}} \sum_{\sigma' \in \{t,f\}^{n-1}} W(t\sigma')}_{E[W \mid x_1 = t] = E[W \mid t]} \right]$$

conditional expectation

$$\mathbb{E}[W] = \frac{1}{2}\left[\mathbb{E}[W|f] + \mathbb{E}[W|t]\right] \leftarrow$$

$$\mathbb{E}[W] \leq \max\{\mathbb{E}[W|f], \mathbb{E}[W|t]\}$$

$\frac{OPT}{2} \leq$ We have a way how to deterministically set X_1 in such a way that when setting all other variables randomly, we still get in expectation a solution of cost $\geq \frac{OPT}{2}$!

More generally

For $b_1, \ldots, b_i \in \{t, f\}$ let

$$\mathbb{E}[W|b_1 b_2 \ldots b_i] = \frac{1}{2^{n-i}} \sum_{\sigma_{n-i} \in \{t,f\}^{n-i}} W(b_1 \ldots b_i \sigma_{n-i})$$

1: $\mathbb{E}[W|b_1 \ldots b_{i-1}] = \frac{1}{2}\left[\mathbb{E}[W|b_1 \ldots b_{i-1} t] + \mathbb{E}[W|b_1 \ldots b_{i-1} f]\right]$

$\Downarrow$

2: $\mathbb{E}[W|b_1 \ldots b_{i-1}] \leq \max\{\mathbb{E}[W|b_1 \ldots b_{i-1} t], \mathbb{E}[W|b_1 \ldots b_{i-1} f]\}$

Note: we can efficiently compute $\mathbb{E}[W|b]$.

ALGORITHM - Derandomized RAND SAT

(1) $i = 1$

(2) if $\mathbb{E}[W|b_1 \ldots b_{i-1} t] \geq \mathbb{E}[W|b_1 \ldots b_{i-1} f]$

set $X_i = t$ otherwise set $X_i := f$

(3) $i := i + 1$, if $i \leq n$ go to (2)

(4) output $b_1 b_2 \ldots b_n$.

THM: the Derandomized RAND SAT is $\frac{1}{2}$-approx. algorithm.

Proof: by 2 applied repeatedly. ∎

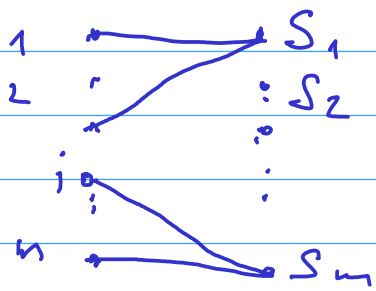
SET COVER

INPUT: sets $S_1, S_2, \dots, S_m \subseteq \{1,2,\dots,n\}$

$c_1, c_2, \dots, c_m \geq 0$

OUTPUT: $I \subseteq \{1,\dots,m\}$ s.t. $\bigcup_{j \in I} S_j = \{1,2,\dots,n\}$

OBJECTIVE: minimize $\sum_{j \in I} c_j$



Parameters: $f = \max_{i \in \{1,\dots,n\}} |\{j \mid i \in S_j\}|$

$\longrightarrow$ $g = \max_{j \in \{1,\dots,m\}} |S_j|$

VERTEX COVER

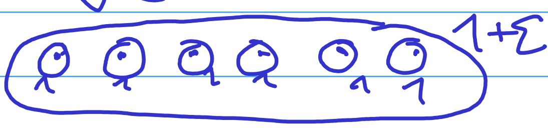
IN: $G = (V,E)$, $c_v \geq 0 \quad \forall v \in V$

OUT: $W \subseteq V$ s.t. $(\forall e \in E)\ e \cap W \neq \emptyset$

Note: $f = 2$ $\quad$ g = max. degree

GREEDY ALGORITHM FOR SC

Ideas: × select the cheapest set
× select the largest set
} does not work



✓ iteratively select the set covering the elements as cheap as possible per element

(1) $I := \emptyset$, $E := \emptyset$
(2) while $E \neq \{1,2,\dots,n\}$ do
  for $j \in \{1,\dots,m\}$ s.t. $S_j \not\subseteq E$, define $p_j = \frac{c_j}{|S_j \setminus E|}$
  $\ell = \operatorname{argmin}_j p_j$
  $I := I \cup \{\ell\}$, $E := E \cup S_\ell$, $\boxed{y_e := p_j}$ $\forall e \in S_j \setminus E$
(3) OUTPUT I

LP Relaxation for SET COVER

(P) $\min \sum_{j=1}^{m} c_j x_j$ for each $S_j \cdots$ binary x_j

$$\sum_{j:\, e \in S_j} x_j \ge 1 \qquad \forall e \in \{1, \dots, n\}$$

$$x_j \ge 0 \qquad j = 1, \dots, m$$

$$\begin{array}{c} \boxed{\quad c \quad} \\ S_1 \;\cdots\; S_m \end{array} \qquad \begin{array}{c} 1 \\ \vdots \\ n \end{array} \boxed{\quad A \quad} \boxed{x} \ge \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

(D) $\max \sum_{e=1}^{n} y_e$

$$\sum_{e \in S_j} y_e \le c_j \qquad \forall j \in \{1, \dots, m\}$$

$$y_e \ge 0 \qquad e = 1, \dots, n$$

$$\boxed{1\ 1 \cdots 1} \qquad \begin{array}{c} S_1 \\ \vdots \\ S_m \end{array} \boxed{\quad A^T \quad} \boxed{y} \le \boxed{c}$$

Define $z' = \frac{1}{H_g} z$ $\qquad H_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$ $\sim \ln n$

z' is a feasible solution of (D).

proof: $z_e \ge 0 \quad \forall e$ ✓

Consider a set S_j: $\qquad \sum_{e \in S_j} z'_e \overset{?}{\le} c_j$

let $k = |S_j|$

order the elements of S_j by their "covering time"

$S_j = \{e_1, e_2, \dots, e_k\}$

e_1 covered last, e_k covered first

Note: for each $e_i \in S_j$ in the iteration in which e_i was covered, it was possible to cover it also by the set S_j at the cost $\le \frac{c_j}{i}$ per element $(1, 2, \dots, i$ NOT YET covered$)$

by greediness: $z_{e_i} \le \frac{c_j}{i}$

$$\sum_{e \in S_j} z'_e = \frac{1}{H_g} \sum_{e \in S_j} z_e \le \frac{1}{H_g} \sum_{i=1}^{k} \frac{c_j}{i} = \frac{1}{H_g} \cdot c_j H_k \overset{k \le g}{\le} c_j \qquad \square$$

THM: GREEDY-SC is H_g-approximation alg.

Proof: by duality of LP & the observation

for a solution I of GREEDY we have

$$\sum_{j \in I} c_j \underset{\text{by def. of } z_e}{=} \sum_{e=1}^{n} z_e = H_g \cdot \sum_{e=1}^{n} z_e' \underset{\text{our}}{\le} H_g \cdot OPT_{LP} \le H_g \cdot OPT$$

where OPT_{LP} is optimal value of LP. ∎