On well quasiordering of finite languages

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Abstract

We investigate here the quasiordering $\preceq$ of finite sets of finite strings over an infinite set of symbols S. We set $\mathcal{K} \preceq \mathcal{L}$ iff it is possible to rename symbols occurring in the strings of $\mathcal{L}$ so that any string of $\mathcal{K}$ is a subsequence of a string of the renamed $\mathcal{L}$. We prove that $\preceq$ is a wqo which answers the question raised by J. Gustedt in [3]. We prove also a stronger version with injective correspondence between strings.

1 Introduction

Strings are finite sequences over S where S is an infinite countable set of *symbols*. *Languages* are finite sets of strings, *babels* are sets of languages. If $A \subseteq S$ then A^* stands for the set of all strings over A. By S^{**} we denote the babel consisting of all languages. We define, for a string $u = a_0 a_1 \ldots a_m$, $S(u) = \bigcup_{i=0}^{m} \{a_i\}$. Similarly $S(\mathcal{L}) = \bigcup_{u \in \mathcal{L}} S(u)$ for any language $\mathcal{L}$.

The notation $u \subset v$ means, for any two sequences $u = a_0 a_1 \ldots a_m$ and $v = b_0 b_1 \ldots b_n$, that u is a subsequence of v: $a_0 = b_{j_0}, a_1 = b_{j_1}, \ldots, a_m = b_{j_m}$ for some m indices $0 \le j_0 < j_1 < \ldots < j_m \le n$. We define, for two languages $\mathcal{L}$ and $\mathcal{K}$, that $\mathcal{L} \le \mathcal{K}$ (via f) iff $u \subset f(u)$ for any $u \in \mathcal{L}$ for some mapping $f : \mathcal{L} \to \mathcal{K}$. A mapping $\varphi : S \to S$ transforms a language $\mathcal{L}$ to the language $\varphi(\mathcal{L}) = \{\varphi(u) \mid u \in \mathcal{L}\}$ where $\varphi(u) = \varphi(a_0 a_1 \dots a_m) = \varphi(a_0)\varphi(a_1)\dots\varphi(a_m)$. We shall investigate the following quasiordering.

Definition 1.1 *$\mathcal{L} \preceq \mathcal{K}$, $\mathcal{L}$ and $\mathcal{K}$ are languages, iff $\mathcal{L} \leq \varphi(\mathcal{K})$ for some $\varphi : S \to S$.*

The above quasiordering was introduced in [7] to generalize *chain minor* ordering of finite posets. We say, in accordance with [7] and with [3], that P is a chain minor of Q (P and Q are finite posets) iff there is a mapping $\rho : Q \to P$ such that any chain in P is isomorphic via ρ to a chain in Q (thus ρ must be onto). Chain minor ordering was introduced in connection with scheduling stochastic project networks [7]. Clearly P is a chain minor of Q iff $\mathcal{L}(P) \preceq \mathcal{L}(Q)$ where $\mathcal{L}(P)$ and $\mathcal{L}(Q)$ are languages consisting of chains in corresponding posets.

By means of that equivalence it has been proven in [3], see also [4], that chain minor is a wqo of finite posets. The proof uses substantially the fact that any "poset language" $\mathcal{L}(P)$ consists of strings without repetitions. The problem whether $\preceq$ is a wqo for languages in general was posed [3]. Generalizing the approach in [3] we answer this question affirmatively.

Theorem 1.2 *$(S^{**}, \preceq)$ is a wqo.*

One can define a stronger quasiordering $\preceq^*$ if the mapping f in the definition of $\preceq$ is injective in addition. We prove that $\preceq^*$ is a wqo as well.

Theorem 1.3 *$(S^{**}, \preceq^*)$ is a wqo.*

In Section 2 we give some preliminaries and demonstrate in a simple case our method. Theorems 1.2 and 1.3 are proven in Sections 3 and 4, respectively. In Section 5 we give counterexamples showing that requiring an injective φ in Definition 1.1 destroyss the wqo property.

2 Absolute minimum about wqo

Any transitive and reflexive binary relation is called a *quasiordering* or, shortly, *qo*. If $(Q, \leq_Q)$ is a qo then $x <_Q y$ means that $x \leq_Q y$ and $y \not\leq_Q x$. A *cone* determined by the element $x \in Q$

is the set $K_x = \{y \in Q \mid y \geq_Q x\}$. A qo $(Q, \leq_Q)$ is a *well quasiordering* or, shortly, *wqo* if it possesses the property characterized by the following lemma. For the proof and for more background we refer to [6].

Lemma 2.1 *Suppose $(Q, \leq_Q)$ is a qo. The following conditions are equivalent.*

1. *For any infinite sequence $(q_i)_{i=0}^{\infty} \subseteq Q$ there are indices $i < j$ such that $q_i \leq_Q q_j$.*
2. *For any infinite sequence $(q_i)_{i=0}^{\infty} \subseteq Q$ there are indices $0 \leq i_0 < i_1 < \ldots$ such that*

$$q_{i_0} \leq_Q q_{i_1} \leq_Q \cdots$$

3. *No infinitely many elements $x_0, x_1, \ldots$ of Q create an antichain or a strictly descending chain*

$$x_0 >_Q x_1 >_Q \ldots$$

Sequences satisfying 1. are called *good*, other sequences are called *bad*. The infinite monotonic subsequence in 2. is called *perfect*. We recall two folcloric but useful statements.

Cone deleting argument. Suppose $(Q, \leq_Q)$ is a wqo and $Q_0, Q_1, \ldots$ are defined by $Q_0 = Q$, $Q_{i+1} = Q_i \backslash K_{q_i}$, $q_i \in Q_i$. Then this sequence is finite, $Q_j = \emptyset$ for some j (otherwise $(q_i)_{i=0}^{\infty} \subseteq Q$ would be a bad sequence).

Product argument. Suppose $(Q_i, \leq_{Q_i})_{i=0}^{r}$ are wqo's, $Q = Q_0 \times Q_1 \times \ldots Q_r$ and $(Q, \leq_{pr})$ is defined by $(x_i)_{i=0}^{r} \leq_{pr} (y_i)_{i=0}^{r}$ iff $x_i \leq_{Q_i} y_i$ for $i = 0, \ldots, r$. Then $(Q, \leq_{pr})$ is a wqo as well (apply Lemma 2.1 $r+1$ times).

Let $(Q, \leq_Q)$ be a qo. The *Higman ordering* $(SEQ(Q), \leq_H)$ on the set

$$SEQ(Q) = \{(I, \ell) \mid I \text{ is a finite linear ordering and } \ell : I \to Q\}$$

of all finite sequences over Q is defined by $(I_0, \ell_0) \leq_H (I_1, \ell_1)$ iff there is an increasing mapping $F : I_0 \to I_1$ such that $\ell_0(x) \leq_Q \ell_1(F(x))$ for any $x \in I_0$. We will use the following classical result of the wqo theory [5].

Theorem 2.2 (Higman) *$(SEQ(Q), \leq_H)$ is a wqo for any wqo $(Q, \leq_Q)$.*

To demonstrate our method in a simple case we prove as an example a weaker version of Higman theorem which deals with the structure $(SET(Q), \leq_S)$ consisting of finite subsets of Q with the qo $A \leq_S B$ iff there is an injective mapping $F : A \to B$ such that $x \leq_Q F(x)$ for any $x \in A$.

Lemma 2.3 *$(SET(Q), \leq_S)$ is a wqo for any wqo $(Q, \leq_Q)$.*

Proof. We prove by a direct argument that any sequence $A = (A_i)_{i=0}^{\infty} \subseteq SET(Q)$ is good to $\leq_S$. We say that $X = (B_i, C_i)_{i=0}^{\infty}$ is a *friend* of A if $(B_i)_{i=0}^{\infty}$ is a subsequence of A, $C_i \subseteq B_i$ for any i, and $(|C_i|)_{i=0}^{\infty}$ is bounded. Set $R(X) = \bigcup_{i=0}^{\infty}(B_i \backslash C_i)$ and $G(i,x) = |K_x \cap (B_i \backslash C_i)|$ where $x \in Q$. We say that X is a *good friend* of A if in addition $\lim_{i\to\infty} G(i,x) = \infty$ (i.e., for any m there is an n such that $i \geq n$ implies $G(i,x) \geq m$) for any fixed $x \in R(X)$.

To prove that any A has a good friend we define a (finite) sequence $X_0, X_1, \ldots$ of friends of A and iniciate it by $X_0 = (A_i, \emptyset)_{i=0}^{\infty}$. Suppose that $X_k = (B_i, C_i)_{i=0}^{\infty}$ is a friend of A which fails to be a good friend: $G(i_0, x), G(i_1, x), \ldots \leq N < \infty$ for some indices $0 \leq i_0 < i_1 < \ldots$ and some $x \in R(X_k)$. Let $D_{i_j} = C_{i_j} \cup (K_x \cap (B_{i_j} \backslash C_{i_j}))$. Then

$$X_{k+1} = (B_{i_j}, D_{i_j})_{j=0}^{\infty}$$

is a friend of A and moreover $R(X_{k+1}) \subseteq R(X_k) \backslash K_x$. According to the cone deleting argument $(X_0, X_1, \ldots)$ terminates in a good friend of A. Notice that when $(|A_i|)_{i=0}^{\infty}$ is bounded then the good friend of A obtained is $(A_{i_j}, A_{i_j})_{j=0}^{\infty}$.

So let $X = (B_i, C_i)_{i=0}^{\infty}$ be a good friend of A. We may assume that $(|C_i|)_{i=0}^{\infty}$ is constant and that $C_0 \leq_S C_1 \leq_S \ldots$ because by the product argument $(C_i)_{i=0}^{\infty}$ contains a perfect subsequence. Take j sufficiently large such that $G(j,x) \geq |B_0 \backslash C_0|$ for any $x \in B_0 \backslash C_0$. As $C_0 \leq_S C_j$ and any $x \in B_0 \backslash C_0$ is majorized (in $\leq_Q$) by sufficiently many elements in $B_j \backslash C_j$ we conclude that $B_0 \leq_S B_j$ — A is good. □

Recall that A^* is the set of all strings over A and that $\subset$ here is the subsequence relation. The following result is an easy and well known consequence of Higman theorem.

Corollary 2.4 *Let A be a finite alphabet. Then $(A^*, \subset)$ is a wqo.*

3 Proof of Theorem 1.2

Any finite collection $G = (E, I) = (E(G), I(G)) = (\{e_i \mid i \in I\}, I)$ of finite sets is called a *set system*, elements of E are called *edges*. We permit repetition of edges and for simplicity we omit the indices of edges when possible. If $H = (F, J)$ is another set system such that $F \subseteq E$ (and $J \subseteq I$) then H is said to be a *subsystem* of G. If E consists of mutually disjoint edges then G is said to be a *disjoint system*.

The *matching number* $M(G)$ of $G = (E, I)$ is defined as the maximum number of edges in a disjoint subsystem of G. A *Q-system* is a couple (G, ℓ) where $\ell : E(G) \to Q$ gives to the edges of G labels from the set Q.

Suppose $A = (G_i, \ell_i)_{i=0}^{\infty}$ is a sequence of Q-systems where $(Q, \leq_Q)$ is a qo. We say that

$$X = (H_i, \ell_i, H_i')_{i=0}^{\infty}$$

is a *friend* of A if $(H_i, \ell_i)_{i=0}^{\infty}$ is a subsequence of A, H_i' is a subsystem of H_i, and $(M(H_i'))_{i=0}^{\infty}$ is bounded.

We define further

$$R(X) = \bigcup_{i=0}^{\infty} \ell_i(E(H_i) \backslash E(H_i')) \subseteq Q \text{ and } G(i, x) = M(H_i''(x))$$

where $x \in Q$ and $H_i''(x)$ is a subsystem of H_i consisting of the edges

$$\{e \in E(H_i) \backslash E(H_i') \mid \ell_i(e) \in K_x\}.$$

We say that X is a *good friend* of A if in addition

$$\lim_{i \to \infty} G(i, x) = \infty$$

for any $x \in R(X)$.

Lemma 3.1 *Any sequence $A = (G_i, \ell_i)_{i=0}^{\infty}$ of Q-systems labelled by a wqo $(Q, \leq_Q)$ has a good friend X.*

Proof. We define again a sequence $X_0, X_1, \ldots$ of friends of A starting with $X_0 = (G_i, \ell_i, \emptyset)_{i=0}^{\infty}$ and show that it terminates in a good friend of A. Suppose $X_k = (H_i, \ell_i, H_i')_{i=0}^{\infty}$ fails to be a good friend of A: $G(i_0, y), G(i_1, y), \ldots \leq N < \infty$ for some indices $0 \leq i_0 < i_1 < \ldots$ and some $y \in R(X_k)$. Then

$$X_{k+1} = (H_{i_j}, \ell_{i_j}, H'_{i_j} \cup H''_{i_j}(y))_{j=0}^{\infty}$$

is clearly a new friend of A and moreover $R(X_{k+1}) \subseteq R(X_k) \backslash K_y$. According to the cone deleting argument after finitely many steps a good friend of A arises. $\square$

Definition 3.2 *A (k,l)-babel where k, l are positive integers is any pair $(\mathcal{B}, A)$ satisfying:*

1. *$\mathcal{B}$ is a babel,*
2. *$A \subseteq S, |A| \leq l$,*
3. *$|S(u) \backslash A| \leq k$ whenever $u \in \mathcal{L}, \mathcal{L} \in \mathcal{B}$.*

Definition 3.3 *We denote by S_A^S, $A \subseteq S$, the set of all mappings $\varphi : S \to S$ such that $\varphi | A = id_A$ and $\varphi^{-1}(A) = A$. For two languages $\mathcal{K}$ and $\mathcal{L}$ the notation $\mathcal{K} \preceq_A \mathcal{L}$ means that $\mathcal{K} \leq \varphi(\mathcal{L})$ for some $\varphi \in S_A^S$.*

Definition 3.4 *Let $\mathcal{L} = (\mathcal{L}_i)_{i=0}^{\infty} \subseteq \mathcal{B}$ be a sequence of languages of a (k,l)-babel $(\mathcal{B}, A)$. Let R be a set of k symbols disjoint to A and let $\chi \in S_A^S$ be fixed such that it maps any $S(u) \backslash A, u \in \mathcal{L}_i, i \geq 0$, injectively to R. We introduce the following sequence of Q-systems $P(\mathcal{L}) = (G_i, \ell_i)_{i=0}^{\infty}$.*

$$I(G_i) = \mathcal{L}_i, E(G_i) = \{S(u) \backslash A \mid u \in \mathcal{L}_i\}, (Q, \leq_Q) = ((R \cup A)^*, \subset),$$

$$\ell_i(e_u) = \chi(u) = \chi(a_0 a_1 \ldots a_m) = \chi(a_0)\chi(a_1) \ldots \chi(a_m).$$

Observation 3.5 *To prove Theorem 1.2. it suffices to prove that $((\mathcal{B}, A), \preceq_A)$ is a wqo for any (k,l)-babel $(\mathcal{B}, A)$.*

Proof. If $\mathcal{L} = (\mathcal{L}_i)_{i=0}^{\infty}$ is a sequence of languages and $|S(u)|, u \in \mathcal{L}_i$, $i \geq 0$, is not universally bounded then $|S(u_0)|$, for some u_0 in some $\mathcal{L}_i$, is at least as big as the sum of lengths of the strings in $\mathcal{L}_0$. Then it is easy to embed the whole $\mathcal{L}_0$ in this single string u_0 and $\mathcal{L}$ is good. Otherwise $|S(u)| \leq c$ for all $u \in \mathcal{L}_i$ and all $i \geq 0$ and hence $\mathcal{L}$ is a $(c, 0)$-babel. $\square$

Lemma 3.6 *$((\mathcal{B}, A), \preceq_A)$ is a wqo for any (k, l)-babel $(\mathcal{B}, A)$.*

Proof. We proceed by double induction on k and l and start with $k = 0$. Then $((\mathcal{B}, A), \preceq_A)$ is a wqo because even $(SET(A^*), \leq_S)$ is a wqo by Lemma 2.3. and Corollary 2.4.

Suppose now that $(\mathcal{B}, A)$ is a (k, l)-babel, $k > 0$, and $\mathcal{L} = (\mathcal{L}_i)_{i=0}^{\infty} \subseteq \mathcal{B}$ is a sequence of languages. We prove that $\mathcal{L}$ is good. We may suppose, renaming appropriately symbols, that $S(\mathcal{L}_i)$ are mutually disjoint up to A and that $S \setminus \bigcup_{i \geq 0} S(\mathcal{L}_i)$ is infinite. Let $P(\mathcal{L}) = (G_i, \ell_i)_{i=0}^{\infty}$ be the sequence defined in Definition 3.4. The labels form a wqo by Corollary 2.4. Thus there is, by Lemma 3.1, a good friend $(H_i, \ell_i, H_i')_{i=0}^{\infty}$ of $P(\mathcal{L})$.

Let F_i be a maximum disjoint subsystem of H_i' and let $U_i = \bigcup E(F_i)$. Clearly $|U_i| \leq ck$ for some constant c (the bound on matching numbers) for any $i \geq 0$. We introduce a set $T, |T| = ck$, of completely new symbols which is disjoint to A and to all $\bigcup E(H_i)$. Let $\rho \in S_A^S$ be such that ρ is an identity on $S \setminus \bigcup_{i \geq 0} U_i$ and maps any U_i injectively to T.

Consider now the babel $\mathcal{C} = (\rho(\mathcal{K}_i))_{i=0}^{\infty}$ where $(\mathcal{K}_i)_{i=0}^{\infty}$ is defined by $\mathcal{K}_i = I(H_i')$. We see that, crucially, $(\mathcal{C}, T \cup A)$ is a $(k-1, ck+l)$-babel because any edge of H_i' must intersect U_i. We may suppose, according to the induction hypothesis, that $\rho(\mathcal{K}_0) \preceq_{A \cup T} \rho(\mathcal{K}_1) \preceq_{A \cup T} \ldots$

We compare the first term to the others: there are mappings $\varphi_i \in S_{A \cup T}^S$ and $f_i : \mathcal{K}_0 \to \mathcal{K}_i, i \geq 1$, such that $\rho(u) \subset \varphi_i(\rho(f_i(u)))$ for any $u \in \mathcal{K}_0$. Let j be such a large number that there are $|E(H_0) \setminus E(H_0')|$ mutually disjoint edges

$$F = \{h_e \mid e \in E(H_0) \setminus E(H_0')\} \subseteq E(H_j) \setminus E(H_j')$$

satisfying $\ell_j(h_e) \supset \ell_0(e)$ for any $e \in E(H_0) \setminus E(H_0')$ and moreover any edge of F is disjoint to $S(f_j(\mathcal{K}_0))$.

We take a mapping $\varphi \in S_A^S$ as follows.

- If $x \in S(f_j(\mathcal{K}_0)) \cap U_j$ then $\rho(y) = \rho(x)$ for at most one $y \in U_0$. If it exists we put $\varphi(x) = y$.
- If $x \in S(f_j(\mathcal{K}_0)) \setminus U_j$ then we put $\varphi(x) = \varphi_j(x)$.
- If $x \in h_e$ for $e \in E(H_0) \setminus E(H_0')$ then $\chi(y) = \chi(x)$ for at most one $y \in e$. If it exists we put $\varphi(x) = y$.

Otherwise φ is defined arbitrarily. Clearly $I(H_0) \leq \varphi(I(H_j))$ and we conclude that the sequence $\mathcal{L}$ is good. $\square$

Lemma 3.6 and Observation 3.5 prove Theorem 1.2.

4 Proof of Theorem 1.3

An easy check shows that only in Observation 3.5. we used the fact that the mapping f of the definition of $\preceq$ had not to be injective. In Lemma 3.6. it has been proven actually that $((\mathcal{B}, A), \preceq_A^*)$ is a wqo for any (k,l)-babel $(\mathcal{B}, A)$. Now we make the whole proof injective by replacing Observation 3.5. by a finer consideration.

Suppose $\mathcal{L} = (\mathcal{L}_i)_{i=0}^\infty \subseteq S^{**}$ is a sequence of languages. We say that $X = (\mathcal{K}_i, \mathcal{K}_i')_{i=0}^\infty$ is a *friend* of $\mathcal{L}$ if $(\mathcal{K}_i)_{i=0}^\infty$ is a subsequence of $\mathcal{L}$, $\mathcal{K}_i' \subseteq \mathcal{K}_i$, $(|\mathcal{K}_i'|)_{i=0}^\infty$ is constant, and $\min\{|S(u)| \mid u \in \mathcal{K}_i'\} \to \infty$ for $i \to \infty$. If moreover $(\max\{|S(u)| \mid u \in \mathcal{K}_i \backslash \mathcal{K}_i'\})_{i=0}^\infty$ is bounded then X is said to be a *good friend* of $\mathcal{L}$.

Consider the following property.

(*) For any c there are in some language $\mathcal{L}_i$ c strings u such that for each of them $|S(u)| \geq c$.

Lemma 4.1 *Suppose $\mathcal{L} = (\mathcal{L}_i)_{i=0}^\infty$ is a sequence of languages not having property (*). Then $\mathcal{L}$ has a good friend.*

Proof. We define then by induction a sequence $X_0, X_1, \ldots$ of friends of $\mathcal{L}$ starting with $X_0 = (\mathcal{L}_i, \emptyset)_{i=0}^\infty$. If $X_k = (\mathcal{K}_i, \mathcal{K}_i')_{i=0}^\infty$ fails to be a good friend of $\mathcal{L}$ then $|S(u_{i_j})| \to \infty$ for $j \to \infty$ for some strings $u_{i_j} \in \mathcal{K}_{i_j} \backslash \mathcal{K}_{i_j}'$ and some indices $0 \leq i_0 < i_1 < \ldots$ Then

$$X_{k+1} = (\mathcal{K}_{i_j}, \mathcal{K}_{i_j}' \cup \{u_{i_j}\})_{j=0}^\infty$$

is a new friend of $\mathcal{L}$. As (*) is violated the growth of $|\mathcal{K}_i'|$ can't proceed arbitrarily long and after finitely many steps a good friend of $\mathcal{L}$ is obtained. $\square$

Proof of Theorem 1.3. Suppose $\mathcal{L} = (\mathcal{L}_i)_{i=0}^\infty \subseteq S^{**}$ is a sequence of languages. If $\mathcal{L}$ has property (*) then $\mathcal{L}_0$ embeds injectively in some $\mathcal{L}_j$. If not then consider a good friend

$X = (\mathcal{K}_i, \mathcal{K}'_i))_{i=0}^{\infty}$ of $\mathcal{L}$. The sequence $(\mathcal{K}_i \backslash \mathcal{K}'_i)_{i=0}^{\infty}$ is a $(c,0)$-babel for some c and by Lemma 3.6 we may suppose it forms a perfect sequence $(\mathcal{K}_0 \backslash \mathcal{K}'_0) \preceq^* (\mathcal{K}_1 \backslash \mathcal{K}'_1) \preceq^* \dots$

So there are mappings $\varphi_i : S \to S$ and injective mappings $f_i : (\mathcal{K}_0 \backslash \mathcal{K}'_0) \to (\mathcal{K}_i \backslash \mathcal{K}'_i), i \geq 1$, such that $u \subset \varphi_i(f_i(u))$ for any $u \in \mathcal{K}_0 \backslash \mathcal{K}'_0$. Now we take such a large j that

$$\min_{u \in \mathcal{K}'_j} |S(u)| \geq \sum_{v \in \mathcal{K}_0 \backslash \mathcal{K}'_0} |S(f_j(v))| + \sum_{v \in \mathcal{K}'_0} length(v).$$

It is easy to extend the injective covering $\mathcal{K}_0 \backslash \mathcal{K}'_0 \preceq^* \mathcal{K}_j \backslash \mathcal{K}'_j$ to the injective covering $\mathcal{K}_0 \preceq^* \mathcal{K}_j$. We conclude that $\mathcal{L}$ is good. $\square$

5 Concluding remarks

Now we show that the fact we did not require an injective φ was crucial to obtain wqo. Let $\mathcal{K} \preceq_* \mathcal{L}$, for two languages $\mathcal{L}$ and $\mathcal{K}$, iff there is an injective $\varphi : S \to S$ such that $\mathcal{K} \leq \varphi(\mathcal{L})$. Consider this example.

Example 5.1 *The infinite babels*

$$\mathcal{B}_0 = \{\{132132\}, \{14213243\}, \{1521324354\}, \{162132435465\}, \dots\}$$

and

$$\mathcal{B}_1 = \{\{ab, bc, ca\}, \{ab, bc, cd, da\}, \{ab, bc, cd, de, ea\}, \dots\}$$

are antichains to $\preceq_*$. Thus $\preceq_*$ is not a wqo.

Note that both babels are antichains also in the ordering obtained by replacing in Definition 1.1. $\mathcal{K} \leq \varphi(\mathcal{L})$ by $\varphi(\mathcal{K}) \leq \mathcal{L}$.

Problem 5.2 *Suppose now that a language $\mathcal{L} = u_0 u_1 \dots u_k$ is a finite sequence of strings rather than just a set and put $\mathcal{L} = u_0 u_1 \dots u_k \preceq \mathcal{K} = v_0 v_1 \dots v_l$ iff there is a mapping $\varphi : S \to S$ and an increasing injection $f : \{0, 1, \dots, k\} \to \{0, 1, \dots, l\}$ such that $u_i \subset \varphi(v_{f(i)})$ for all $i = 0, 1, \dots, k$. Is this $\preceq$ stil a wqo ?*

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References

[1] *Graph minors* (N. Robertson and P. Seymour, editors), Proceedings of the AMS-IMS-SIAM Joint Summer Research Conference, Seattle WA, June 1991.

[2] *Graphs and Orders* (I. Rival, ed.), D.Reidel Publishing Company, Dordrecht 1985.

[3] J. GUSTEDT, Algorithmic aspects of ordered structures, PhD thesis, Technische Universität Berlin 1992.

[4] J. GUSTEDT, Well-quasi-ordering finite posets, in [1].

[5] G. HIGMAN, Ordering by divisibility in abstract algebras, *Proc. London Math. Society* **2** (1952), 326–336.

[6] E. C. MILNER, Basic wqo and bqo theory, in [2].

[7] R. H. MÖHRING AND R. MÜLLER, A combinatorial approach to obtain bounds for stochastic project networks, Technical report, Tech. Un. Berlin 1991.