Last MA 2 tutorial test, January 11, 2024

For 90 minutes and each problem is worth 4 points. At least 50% of the points (= 16 pts.) are required for passing. Justify your answers.

1. Define metric spaces (M, d). Show that for every $a, b, c, e \in M$ it holds that $d(a, e) \le d(a, b) + d(b, c) + d(c, e)$.
2. Define compact sets in metric spaces (M, d). Is it true that the whole set M is always compact?
3. Is it true that the identity map $f\colon M \to M$, $f(x) = x$, from a metric space (M, d) to itself is always continuous?
4. Let $A \in \mathbb{R}^{k\times l}$ and $B \in \mathbb{R}^{l\times m}$ be two real matrices. Write the formula for their product $A \cdot B$.
5. Is the real axis $\mathbb{R}$ (with the metric $d(x, y) = |x - y|$) a complete metric space?
6. Compute for the function $f(x, y, z) = \mathrm{e}^{x+y}\sin(x + y + z)$ partial derivatives f_x, f_y and f_z.
7. Why can we be sure that the function $f(x, y) = x^2 - y\sin(xy)$ attains on the set $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 1\}$ (the closed unit disc in the plane) its maximum value? Give a simple reason.
8. And what about the function $f(x, y) = x^2 + y^2$ and the set $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ (the open unit disc in the plane)?