A generating functions proof of this partition identity: the sum over all partitions of a number n with mutually distinct parts, each partition weighted by $(-1)^{k+1}m$ where k is the number of parts and m the smallest part, equals to the number of divisors of n

Martin Klazar[1]

May 12, 2012

In the middle of the article of Andrews and Freitas [1, p. 146] one finds, along with a great number of more sophisticated identities, the (formal power series) identity

$$\sum_{n=0}^{\infty}\Big(1-\prod_{k=n+1}^{\infty}(1-q^k)\Big)=\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}. \qquad (1)$$

The right side equals, after expansion of geometric series, to $\sum_{n\ge 1}\tau(n)q^n$ where $\tau(n)$ denotes the number of divisors of n. It is easy to interpret combinatorially the coefficient of q^n on the left side and thus we get the partition identity stated in the title:

$$\sum_{0<\lambda_1<\cdots<\lambda_k,\sum\lambda_i=n}(-1)^{k+1}\lambda_1=\tau(n)$$

(λ_i are integers). The right side in fact counts partitions of n with all parts equal each to the other. For example, the 15 partitions of $n=12$ into distinct parts,

$$(12),(11,1),(10,2),93,921,84,831,75,741,732,651,642,6321,543 \text{ and } 5421,$$

generate the sum $12-1-2-3+1-4+1-5+1+2+1+2-1+3-1=6$, which indeed counts the six divisors $1,2,3,4,6$ and 12 of 12.

A hint for a proof of (1) is given in [1, p. 146], to apply [1, Lemma 2.2 on p. 140] to certain functions $f(x)$ and $g(x)\equiv 1$ (we mention $f(x)$ below) and then let $x\to 1$, which seems to indicate an analytic proof. We could not understand the hint as it seems that the purported application of Lemma 2.2 only returns useless tautology that if $f(x)=\sum_{n\ge 0}f_nx^n$ then $f(x)=\sum_{n\ge 0}f_nx^n$; it seems to us that Lemma 2.2 is completely irrelevant to the proof, certainly to the one we are going to present. Nevertheless, the identity is beautiful, both combinatorially and for power series, and worth an effort. Our aim was to work out for ourselves a rigorous, purely formal proof of (1). Even though we will substitute 1 for the formal variable x, the proof we give is still purely formal in the sense that it only uses formal convergence of power series, i.e., a non-archimedean norm, as opposed to analytic proofs that work with archimedean norms.

Let

$$c_n=c_n(q)=\frac{1}{(q)_n}=\frac{1}{(1-q)(1-q^2)\dots(1-q^n)},\ c_0=1.$$

[1]klazar@kam.mff.cuni.cz

Thus, clearly,
$$(1-q^n)c_n = c_{n-1},\ n \geq 1.$$
From this trivial (but crucial—everything stems from it) identity it follows easily by induction that
$$c_n = \frac{1 + qc_1 + q^2c_2 + \cdots + q^{n-1}c_{n-1}}{1-q^n}. \tag{2}$$
Now we make a longer leap—we claim that
$$\sum_{n=0}^{\infty} \frac{x^n}{(q)_n} = \frac{1}{(x)_\infty} \tag{3}$$
where
$$(a)_n = (1-a)(1-aq)\dots(1-aq^{n-1}) \text{ and } (a)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$
Indeed, the right side $F(x,q)$ satisfies the functional equation
$$F(x,q) = (1-x)^{-1}F(xq,q) = (1+x+x^2+\dots)F(xq,q)$$
and so if $F(x,q) = \sum_{n\geq 0} d_n x^n$ then the coefficients d_n are forced to satisfy the recurrence relation
$$d_n = q^n d_n + q^{n-1}d_{n-1} + \cdots + qd_1 + d_0.$$
Since it coincides with (2) and $d_0 = c_0 = 1$, we see that $d_n = c_n$ and the identity (3) follows. (It is the simplest form of the q-binomial theorem.) A simple rearrangement gives
$$\sum_{n=0}^{\infty}\Big(1 - \frac{(q)_\infty}{(q)_n}\Big)x^n = \frac{1}{1-x} - \frac{(q)_\infty}{(x)_\infty} \tag{4}$$
—on the right side we get the function $f(x)$ of [1]. For $x = 1$ the left side of (4) gives the left side of (1) and it remains to show that for $x = 1$ the right side of (4) produces the right side of (1). In the present form its value for $x = 1$ is undefined.

We rigorously justify cancellation of the factor $1-x$ on the right side of (4). The identities (4) and (3) live in $(\mathbb{C}[[q]])[[x]]$, the ring of formal power series in x whose coefficients are formal power series in q with complex coefficients. We consider formal convergence in $\mathbb{C}[[q]]$, i.e., the non-archimedean metric coming from the norm $\|f(q)\| = 2^{-\mathrm{ord}(f)}$ (where $\mathrm{ord}(f)$ is the smallest n such that q^n has in $f(q)$ nonzero coefficient and $\mathrm{ord}(0) = +\infty$). We consider the subring R of $(\mathbb{C}[[q]])[[x]]$,
$$R = \{\textstyle\sum_{n\geq 0} a_n x_n \in (\mathbb{C}[[q]])[[x]] \mid \|a_n\| \to 0, n \to \infty\}.$$

This is indeed a subring and even differential one—if $f \in R$ then all derivatives $f', f'', \dots$ (with respect to x) remain in R. Note that the power series in (3) is not in R while that in (4) lies in R; this is the important effect of the simple rearrangement. We have a ring homomorphism from R to $\mathbb{C}[[q]]$,

$$f(x) = a_0 + a_1x + a_2x^2 + \dots \mapsto f(1) = a_0 + a_1 + a_2 + \dots;$$

this is well defined since the series $a_0 + a_1 + a_2 + \dots$ formally converges in $\mathbb{C}[[q]]$.

Once we realize this, the following claim is expected but for completeness we prove it.

Claim. *Any $f(x) \in R$ such that $f(1) = 0$ has factorization*

$$f(x) = (x-1)g(x)$$

where $g(x) \in R$ and $g(1) = f'(1)$.

Proof. Let $f(x) = a_0 + a_1x + a_2x^2 + \dots$. Then

$$\begin{aligned} f(x) &= f(x) - f(1) \\ &= (a_0 + a_1x + a_2x^2 + a_3x^3 + \dots) - (a_0 + a_1 + a_2 + a_3 + \dots) \\ &= a_1(x-1) + a_2(x^2-1) + a_3(x^3-1) + \dots \\ &= (x-1)(a_1 + a_2(x+1) + a_3(x^2+x+1) + \dots) \\ &= (x-1)g(x) \end{aligned}$$

where the infinite series sum due to formal convergence in $\mathbb{C}[[q]]$ and $g(x) = \sum_{n\geq 0}(\sum_{m>n} a_m)x^n$. It is easy to see that $g(x) \in R$ and differentiating $f(x) = (x-1)g(x)$ and setting $x = 1$ we get $f'(1) = g(1)$. $\square$

In fact, one has in R the full Taylor series expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!}$$

but we need only the beginning $f(x) = f(1) + f'(1)(x-1) + O((x-1)^2)$.

Using the claim we can calculate the value of the power series in (4) for $x = 1$ also by means of the right side. The right side can be written as $(1-x)^{-1}h(x)$ for some $h(x) \in R$ with $h(1) = 0$, and so by the claim the power series in (4) for $x = 1$ also equals to $-h'(1)$. Namely, it equals to

$$\left(\frac{1}{1-x}\left(1 - \frac{(q)_\infty}{(xq)_\infty}\right)\right)_{x=1} = -\left(1 - \frac{(q)_\infty}{(xq)_\infty}\right)'_{x=1} = \left(\frac{(q)_\infty}{(xq)_\infty}\right)'_{x=1}.$$

We set

$$S_n = S_n(x) = \frac{1-q^n}{1-xq^n}.$$

Leibniz rule for infinite products gives

$$\left(\frac{(q)_\infty}{(xq)_\infty}\right)'_{x=1} = \left(\prod_{n=1}^{\infty} S_n\right)'_{x=1} = \left(\prod_{n=1}^{\infty} S_n \sum_{k=1}^{\infty} \frac{S'_k}{S_k}\right)_{x=1} = \sum_{k=1}^{\infty} S'_k(1)$$

as $S_n(1) = 1$. Since

$$S'_k(1) = \frac{q^k}{1-q^k},$$

we are done—we proved that the power series in (4) for $x = 1$ equals also to the right side of (1) and thus we established identity (1).

References

[1] G. E. Andrews and P. Freitas, Extension of Abel's lemma with q-series implications, *Ramanujan J.* **10** (2005), 137–152.