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On the recent result of Green and Tao about arbitrarily long arithmetic progressions of primes

B. Green (1977) & T. Tao (1975), Apr 8, 2004, Arxiv: $\forall k\ \exists$ primes $p_1 < p_2 < \dots < p_k$ s.t.

$$p_2 - p_1 = p_3 - p_2 = \dots = p_k - p_{k-1} \quad (= \Delta)$$

Related results

Dirichlet (1837): $\forall\, a, m \in \mathbb{N}$, $(a,m) = 1\ \exists\, p \equiv a \pmod m$

Шнирельман (1930): $\exists\, k$ s.t. $n = p_1 + p_2 + \dots + p_\ell$, $\ell \le k$, has solution $\forall\, n \geq 2$.

Виноградов (1937): $2n+1 = p_1 + p_2 + p_3$ has solution $\forall\, n > n_0$.

van der Corput (1939): $\exists\, \infty$ many triples

$$p_1 \overset{\Delta}{\longleftrightarrow} p_2 \overset{\Delta}{\longleftrightarrow} p_3$$

Heath-Brown (1981): $\exists\, \infty$ many quadruples

$$n_1 \overset{\Delta}{\longleftrightarrow} p_2 \overset{\Delta}{\longleftrightarrow} p_3 \overset{\Delta}{\longleftrightarrow} n_4$$

where $\Omega(n_1 n_4) \leq 3$.

Szemerédi's theorem (1975)

$\forall \delta > 0,\ k \geq 3\ \exists N_0(\delta, k)$: $N \geq N_0(\delta, k)$ & $X \subset \mathbb{Z}_N$ with $|X| \geq \delta N$, then $X \supset$ A.P. of length k.

[$\mathbb{Z}_N$ is $\mathbb{Z}/N\mathbb{Z}$, N big prime]

The proof of Green and Tao consists of 2 steps

① Szemerédi's thm. $\Longrightarrow$ Relative Sz. thm.

Ergodic-combinatorial part

RST: $\delta > 0$, $k \geq 3$, $\tilde{X} \subset \mathbb{Z}_N$ pseudorandom, $X \subset \tilde{X}$, $\frac{|X|}{|\tilde{X}|} \geq \delta$.

Then $X \supset$ A.P. of length k, for N big enough.

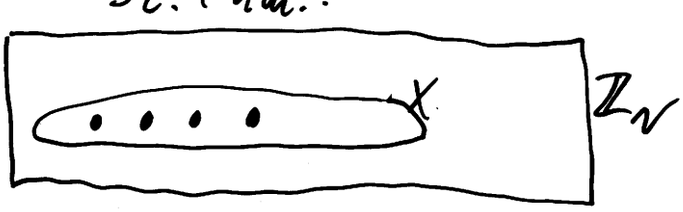
② Number-theoretic part

$\mathbb{Z}_N \supset \mathcal{P}$ - primes ($< N$). $\exists$ pseudor. $\tilde{\mathcal{P}} \supset \mathcal{P}$ s.t. $\frac{|\mathcal{P}|}{|\tilde{\mathcal{P}}|} \geq \delta$.

$\tilde{\mathcal{P}} \subset \mathbb{Z}_N$

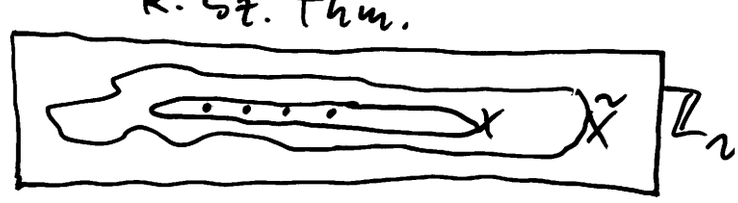
① & ② $\Longrightarrow$ $\mathcal{P} \supset$ A.P. of length k for $\forall k$.

Sz. Thm.:



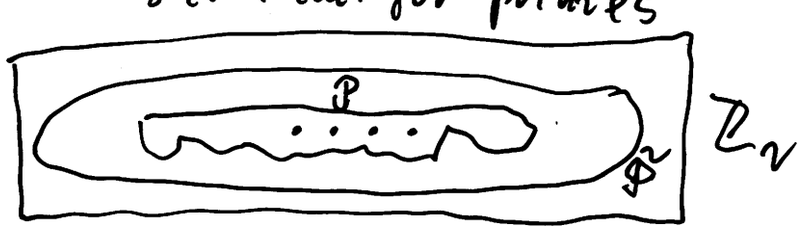
$|X|/N \geq \delta$

R. Sz. thm.



$|X|/|\tilde{X}| \geq \delta$

Sz. thm. for primes



$|\mathcal{P}|/|\tilde{\mathcal{P}}| \geq \delta$

<u>In more details...</u> N – large prime, $o(1)$, $O(1)$ – w.r.t. $N \to \infty$.

1) Relative Szemerédi's theorem

A - finite set, $f: A \to \mathbb{R}$, $\mathbb{E}(f) = \mathbb{E}(f(x) \mid x \in A) :=$
$:= \frac{1}{|A|} \sum_{x \in A} f(x)$. $\nu: \mathbb{Z}_N \to \mathbb{R}^+$ measure: $\mathbb{E}(\nu) = 1 + o(1)$

Thm. 2.3

<u>Szemerédi's thm.</u>: $0 < \delta \leq 1$, $k \geq 3$, $f: \mathbb{Z}_N \to \mathbb{R}^+$ s.t. $0 \leq f(x) \leq 1$ $\forall x \in \mathbb{Z}_N$ & $\mathbb{E}(f) \geq \delta$.

Then $\mathbb{E}\left(\prod_{i=0}^{k-1} f(x+ir) \mid x, r \in \mathbb{Z}_N\right) \geq \underbrace{c(k,\delta)}_{>0} - o(1)$.

<u>Relative Szemerédi's thm. (Thm. 3.5)</u>: $0 < \delta \leq 1$, $k \geq 3$, $\nu: \mathbb{Z}_N \to \mathbb{R}^+$ k-pseudor. measure, $f: \mathbb{Z}_N \to \mathbb{R}^+$ s.t. $0 \leq f(x) \leq \nu(x)$ $\forall x \in \mathbb{Z}_N$ & $\mathbb{E}(f) \geq \delta$.

Then $\mathbb{E}\left(\prod_{i=0}^{k-1} f(x+ir) \mid x, r \in \mathbb{Z}_N\right) \geq c(k,\delta) - o(1)$.

[$c(k,\delta) > 0$ the same in both thm's. $o(1) = o_{k,\delta}(1)$.]

So constant 1 was "just" replaced by any k-pseudorandom measure $\nu(x)$.

What is k-pseudorandomness?

A measure $\nu : \mathbb{Z}_N \to \mathbb{R}^+$ is <u>k-pseudorandom</u> if it satisfies 2 conditions.

a) (lin. forms condition)

$$L = \underbrace{\boxed{\in \mathbb{Q}^{m \times t}}}_{\underline{t \le 3k-4}} \updownarrow \underline{m \le k 2^{k-1}}, \quad L_{ij} = \frac{a}{b} \in \mathbb{Q}, \quad \underline{|a|, |b| \le k}$$

$\in \mathbb{Z}_N$

$b_1, b_2, \dots, b_m \in \mathbb{Z}_N$

$$\psi_i(\bar{x}) = \sum_{j=1}^{t} L_{ij} x_j + b_i, \quad i = 1, 2, \dots, m.$$

L nondegenerate (no 0-row, no row is a multiple of another row). Then

$$\mathbb{E}\left(\prod_{i=1}^{m} \nu(\psi_i(\bar{x})) \,\middle|\, \bar{x} \in \mathbb{Z}_N^t\right) = 1 + o_k(1).$$

b) (correlation condition)

$\forall m$, $\underline{m \le 2^{k-1}}$, $\exists$ weight function $\tau_m : \mathbb{Z}_N \to \mathbb{R}^+$ s.t.

$\mathbb{E}(\tau_m^q) = O_{k,q}(1)$ $\forall q \in \mathbb{N}$ and

$$\mathbb{E}\left(\prod_{i=1}^{m} \nu(x + h_i) \,\middle|\, x \in \mathbb{Z}_N\right) \le \sum_{1 \le i < j \le m} \tau_m(h_i - h_j)$$

$\forall$ m-tuple $h_1, h_2, \dots, h_m \in \mathbb{Z}_N$ (not necessarily distinct).

[In b) $L = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ is degenerate, $b_i = h_i$.]

(2) Enveloping primes by a pseudor. set; with >0 relative density of primes.

$\Lambda : \mathbb{N}, \mathbb{Z}_N \to \mathbb{R}^+$, $\Lambda(n) = \begin{cases} \log p & \dots\ n = p^r \\ 0 & \dots\ n \neq p^r \end{cases}$

von Mangoldt function

[Case $r \geq 2$ will be neglected]

Recall that $\Lambda(n) = \sum_{d|n} \mu(d) \log \frac{n}{d}$

[Möbius inverse of $\log n = \sum_{d|n} \Lambda(d)$, μ is M. function,

$\mu(n) = \begin{cases} (-1)^r & \dots\ n = p_1 p_2 \dots p_r \\ 0 & \dots\ n \text{ is not } \square\text{-free} \end{cases}$]

PNT $\to$ $\mathbb{E}(\Lambda) = 1 + o(1)$ and Λ is supported by primes (we can neglect p^r, $r \geq 2$).

Suppose we have a k-pseudor. measure ν s.t. $\nu(n) \geq c(q) \cdot \Lambda(n)$ $\forall n \in \mathbb{Z}_N$. Then we have the main result. Unfortunately, no such ν exists.

[$q \in \mathbb{N}$, Λ concentrated only on those a mod q classes, for which $(a,q) = 1$ but ν must be evenly distributed among all q mod q classes. $\varphi(q)$ And $\liminf_{q \to \infty} \frac{\varphi(q)}{q} = 0$.] G.-T. get around this by the W-trick

$w = w(N) \to \infty$ slowly with $N \to \infty$.

$$W = W(w) = \prod_{p \le w(N)} p \quad \Big| \quad \tilde{\Lambda}(n) = \begin{cases} \frac{\varphi(W)}{W} \log(Wn+1) & \dots \text{ if } Wn+1 \text{ is prime} \\ 0 \dots \text{ otherwise} \end{cases}$$

A.P. in n's $\rightsquigarrow$ A.P. in the values $Wn+1$.

<u>Enveloping primes</u> (Prop. 8.1.) $\varepsilon_k = \frac{1}{2^k (k+4)!}$,

N prime and big. Then $\exists$ k-pseudor. measure ν: $: \mathbb{Z}_N \to \mathbb{R}^+$ s.t.

$$\nu(n) \geq \frac{1}{k\, 2^{k+5}} \tilde{\Lambda}(n) \text{ holds for}$$

every $n \in [\varepsilon_k N, 2\varepsilon_k N]$.

<u>Proof of the main result.</u>

Set $f(n) = \begin{cases} k^{-1} 2^{-k-5} \tilde{\Lambda}(n) \dots & n \in [\varepsilon_k N, 2\varepsilon_k N] \\ 0 \dots \text{ else.} \end{cases}$

$$\mathbb{E}(f) = N^{-1} k^{-1} 2^{-k-5} \sum_{n \in [\dots]} \tilde{\Lambda}(n) = \dots = \frac{\varepsilon_k}{k\, 2^{k+5}} (1 + o(1)) = \delta > 0$$

[by PNT for ar. progressions]. By Prop. 8.1 and Thm. 3.5, $\mathbb{E}\left(\prod_{i=0}^{k-1} f(x+iv) \mid x, v \in \mathbb{Z}_N\right) \geq c(k, \delta) - o(1)$

> 0 for big N — we get a genuine A.P. of length k of primes in $\mathbb{N}$.

[$\varepsilon_k < \frac{1}{k}$. Also, case $v = 0$ contributes only $O\left(\frac{\log^k N}{N}\right) = o(1)$.]

How is the k-pseudorandom measure $\nu(n)$ that majorizes $c(q)\cdot\tilde{\Lambda}(n)$ defined?

$$\Lambda(n) = \sum_{d|n} \mu(d) \log \frac{n}{d} = \sum_{d|n} \mu(d) \log_+ \frac{n}{d}$$ where

$\log_+ x = \max(\log x, 0)$.

$R>0$ parameter [$R = N^{\text{small exp.}}$]

$$\Lambda_R(n) := \sum_{\substack{d|n \\ d\le R}} \mu(d) \log \frac{R}{d} = \sum_{d|n} \mu(d) \log_+ \frac{R}{d}.$$

(Goldston - Yildirim)

Definition of ν (Def. 8.3) $R := N^{k^{-1}2^{-k-4}}$ and ε_k as before ($\varepsilon_k = \frac{1}{2^k (k+4)!}$). We define $\nu : \mathbb{Z}_N \to \mathbb{R}^+$ by

$$\nu(n) = \begin{cases} \frac{\varphi(W)}{W} \cdot \frac{1}{\log R} \cdot \Lambda_R(Wn+1)^2 \dots & n \in [\varepsilon_k N, 2\varepsilon_k N] \\ 1 \dots & \text{otherwise} \end{cases}$$

It is shown that $\nu(n) \ge \frac{1}{k 2^{k+5}} \tilde{\Lambda}(n)$ (Lemma 8.4), $\mathbb{E}(\nu) = 1 + o(1)$ (Lemma 8.7) and that ν is k-pseudorandom (Prop. 8.8 and Prop. 8.10). The last two results are based ~~on~~ on asymptotics for expressions involving $\Lambda_R(n)$, derived by method due to Goldston and Yildirim.

Further details

(1) Relative Szemerédi's theorem (Thm. 3.5)

Its proof uses 3 results: Szemerédi's thm. (Thm. 2.3), Generalized von Neumann thm. (Prop. 5.3) and Generalized Koopman-von N. thm. (Prop. 7.1).

$f: \mathbb{Z}_N \to \mathbb{R}$, $\|f\|_{U^d}$ - Gowers' uniformity norm

Generalized von N. thm. (Prop. 5.3)

$\nu: \mathbb{Z}_N \to \mathbb{R}^+$ k-pseudor. measure, $f_0, \dots, f_{k-1}: \mathbb{Z}_N \to \mathbb{R}$ are s.t. $|f_i(x)| \leq \nu(x) + 1$ $\forall x \in \mathbb{Z}_N$, $i = 0 \dots k-1$.

$[\Leftrightarrow \leq \nu(x)]$

Then $\mathbb{E}\left(\prod_{i=0}^{k-1} f_i(x+ir) \mid x, r \in \mathbb{Z}_N\right) = O\left(\min_{i=0,\dots,k-1} \|f_i\|_{U^{k-1}}\right) + o(1)$.

Definition of $\|\cdot\|_{U^d}$.

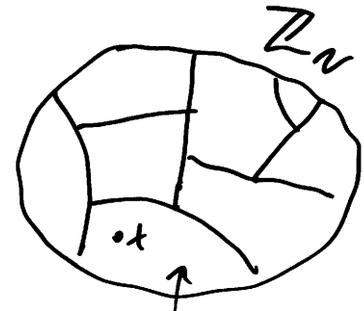
$f: \mathbb{Z}_N \to \mathbb{R}$, $\omega \in Q_d = \{0,1\}^d$, $|\omega| = \omega_1 + \dots + \omega_d$. For $h \in \mathbb{Z}_N^d$ we set $\omega \cdot h = \omega_1 h_1 + \dots + \omega_d h_d$.

$[\{x + \omega \cdot h \mid \omega \in Q_d\}$ is d-dim cube in $\mathbb{Z}_N$.$]$

$$\|f\|_{U^d} := \left(\mathbb{E}\left(\prod_{\omega \in Q_d} f(x + \omega \cdot h) \mid x \in \mathbb{Z}_N, h \in \mathbb{Z}_N^d\right)\right)^{1/2^d}$$

Note that $\|f\|_{U^1} = (\mathbb{E}(f)^2)^{1/2} = |\mathbb{E}(f)|$ is just semi norm. For $d \geq 2$ $\|\cdot\|_{U^d}$ is a norm.

σ-algebra $\mathcal{B}$ in $\mathbb{Z}_N$: $\mathcal{B} \subset \exp(\mathbb{Z}_N)$ closed to $\cup, \cap, \setminus$; $\emptyset, \mathbb{Z}_N \in \mathcal{B}$. Atoms of $\mathcal{B}$: minimal $\neq \emptyset$ elements; form a partition of $\mathbb{Z}_N$:



$A(x)$ - atom of $\mathcal{B}$ containing x.

$f: \mathbb{Z}_N \to \mathbb{R}$, $\mathbb{E}(f|\mathcal{B}): \mathbb{Z}_N \to \mathbb{R}$

given by $\mathbb{E}(f|\mathcal{B})(x) = \frac{1}{|A(x)|} \sum_{y \in A(x)} f(y)$

- Conditional expectation of f w.r.t. $\mathcal{B}$.

$\Omega \subset \mathbb{Z}_N$, $\chi_\Omega(x) = \begin{cases} 1 \dots x \in \Omega \\ 0 \dots x \notin \Omega. \end{cases}$

<u>Generalized Koopman-von N. thm. (Prop. 7.1)</u>

$\nu: \mathbb{Z}_N \to \mathbb{R}^+$ k-pseudor. measure, $f: \mathbb{Z}_N \to \mathbb{R}$ s.t. $0 \le f(x) \le \nu(x)$ $\forall x \in \mathbb{Z}_N$, $0 < \varepsilon < 1$, N big (dep. on ε).

Then $\exists$ σ-alg. $\mathcal{B}$, $\Omega \in \mathcal{B}$ s.t.

$\mathbb{E}(\nu \cdot \chi_\Omega) = o_\varepsilon(1)$ ... i.e., Ω is small (exceptional) set

$\|(1-\chi_\Omega)\mathbb{E}(\nu - 1|\mathcal{B})\|_{L^\infty} = o_\varepsilon(1)$ ... i.e., ~~~~.

$\|(1-\chi_\Omega)(f - \mathbb{E}(f|\mathcal{B}))\|_{U^{k-1}} \le \varepsilon^{1/2^k}$ .... i.e., ~~~~.

Proof of Prop. 7.1 takes cca 12 pages and uses, e.g., Weierstrass approximation thm. Proof of Prop. 5.3 takes cca 4 pages and uses Cauchy-Schwarz ineq. and k-pseudorandomness of ν.

Thm. 2.3 is a black box (proof takes 50, resp. 20, pages elsewhere).

To prove Relative Sz. thm. (Thm. 3.5) is now easy (1/2 page). We have f ~~assim~~ and ν as in thm. 3.5. Take $\mathcal{B}$ and Ω from Prop. 7.1 and write

$$f = \underbrace{(f - \mathbb{E}(f \mid \mathcal{B}))}_{g} + \underbrace{\mathbb{E}(f \mid \mathcal{B})}_{h}.$$

Express

$$\mathbb{E}\Big(\prod_{i=0}^{q-1} \underbrace{f(x+ir)}_{=g(\sim)+h(\sim)} \,\Big|\, x, r \in \mathbb{Z}_N\Big)$$

as the main term + error:

$$= \underbrace{\mathbb{E}\Big(\prod_{i=0}^{q-1} h(x+ir) \,\Big|\, \text{-}\|\text{-}\Big)}_{M} + \underbrace{\sum_{2^q-1 \text{ terms}} \mathbb{E}\Big(\prod_{i=0}^{q-1} \begin{matrix} g \\ h \end{matrix}(x+ir) \,\Big|\, \text{-}\|\text{-}\Big)}_{E}$$

at least once g

We work outside the negligible Ω. $M \geq c(q,\delta) > 0 - o_\varepsilon(1)$ by Sz. thm. (Thm. 2.3). (We "contract" atoms outside Ω, $\mathbb{E}(\nu \mid \mathcal{B})$ behaves like $\equiv 1$.)

$E = O(\varepsilon^{1/2^q})$ by Generalized von N. thm. (Prop. 5.3).

So $\mathbb{E}(\sim) \geq c(q,\delta) - O(\varepsilon^{1/2^q}) - o_\varepsilon(1)$.

② Enveloping primes

We need to prove that $\nu(n) = \begin{cases} \frac{\varphi(W)}{W} \cdot \frac{\Lambda_R(Wn+1)^2}{\log R} & \\ \dots \text{ for } n \in [\varepsilon_q N, 2\varepsilon_q N] \\ 1 \dots \text{ otherwise} \end{cases}$

is q-pseudorandom ($R = N^{q^{-1}2^{-q-4}}$, $\Lambda_R(n) = \sum_{d|n} \mu(d) \log_+ \frac{R}{d}$, $\varepsilon_q = \frac{1}{2^q (q+4)!}$).

The lin. forms condition on ν is proved by the following asymptotics due to Goldston and Yildirim.

Prop. 8.5

$L = \boxed{\in \mathbb{Z}^{m \times t}}$ $\updownarrow m$, $\leftrightarrow t$ - nondegenerate matrix (cf. l.f. condition) $|L_{ij}| \le \frac{1}{2}\sqrt{w(N)}$, $b_1, \dots, b_m \in \mathbb{Z}$

$$\psi_i(\bar{x}) = \sum_{j=1}^{t} L_{ij} x_j + b_i, \quad i = 1 \dots m$$

$B = I_1 \times I_2 \times \dots \times I_t \subset \mathbb{R}^t$, $|I_j| \ge R^{10m}$ $\forall j$. (intervals) Then (if $w(N) \to \infty$ as $N \to \infty$ slowly enough)

$$\mathbb{E}\left(\prod_{i=1}^{m} \Lambda_R(W\psi_i(\bar{x})+1)^2 \,\middle|\, \bar{x} \in B \cap \mathbb{Z}^t \right) = (1+o_m(1)) \left(\frac{W \log R}{\varphi(W)} \right)^m.$$

Similar Prop. 8.6 takes care of the correlation condition; it is proved by methods due to Goldston and Yildirim as well.

Let us review the proof of Prop. 8.5.

$B = \prod_{j=1}^{t} I_j \subset \mathbb{R}^t$, $|I_j| \geq R^{10m}$. We seek the asymptotics of

$$\mathbb{E}\Big(\prod_{i=1}^{m} \Lambda_R(W\psi_i(\bar{x})+1)^2 \,\Big|\, \bar{x} \in B \cap \mathbb{Z}^t\Big). \qquad (*)$$

$(*)$ is transformed (up to a small error) in the integral form

$$\boxed{\frac{1}{(2\pi i)^m} \overbrace{\int_{\Gamma_1} \cdots \int_{\Gamma_1}}^{2m} F(z,u) \prod_{j=1}^{m} \frac{R^{z_j+u_j}}{z_j^2\, u_j^2}\, dz_j\, du_j} \qquad (**)$$

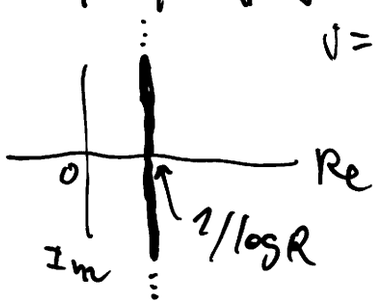
where

$z = z_1, \dots, z_m \in \mathbb{C}^m$; $u = u_1, \dots, u_m \in \mathbb{C}^m$; $d = d_1, \dots, d_m \in \mathbb{N}^m$; $e = e_1, \dots, e_m \in \mathbb{N}^m$

$$F(z,u) = \sum_{\substack{d_1,\dots,d_m \\ e_1,\dots,e_m \in \mathbb{N}}} \Big(\prod_{j=1}^{m} \frac{\mu(d_j)\mu(e_j)}{d_j^{z_j}\, e_j^{u_j}}\Big) \prod_p M(p,d,e)$$

$$M(p,d,e) = \mathbb{E}\Big(\prod_{\substack{j=1 \\ p \nmid d_j e_j}}^{m} \chi_{W\psi_j(\bar{x})+1 \equiv 0 \bmod p} \,\Big|\, \bar{x} \in \mathbb{Z}_p^t\Big) =$$

$$= \frac{1}{|\mathbb{Z}_p^t|} \#\{\bar{x} \in \mathbb{Z}_p^t : p \nmid d_j e_j \Rightarrow W\psi_j(\bar{x})+1 \equiv 0 \bmod p\}_{j=1\dots m},$$

and $\Gamma_1 =$ 

. This is achieved by using definition of Λ_R, changing summation order and using the formula $\frac{1}{2\pi i}\int_{\Gamma_1} \frac{x^z}{z^2}\, dz = \log_+ x$ (known from Perron's Σ or theory of $\zeta(s)$).

$D_\sigma^m = \{z_j, u_j : -\sigma < \mathrm{Re}(z_j), \mathrm{Re}(u_j) < 100,\ j = 1 \dots m\}$, $\sigma > 0$, for $G = G(z,u)$ holom. in $\underset{D}{\underset{\parallel}{D_\sigma^m}}$ we set $V(G, k, D)$ to be the largest $\sup_{z,u \in D} |\bar{G}(z,u)|$ where $\bar{G}$ = some ∂G of order $\leq k$.

$F = F(z,u)$, appearing in (**), factorizes as $F = G_1 G_2 G_3$ where $G_3(z,u) = \prod_{j=1}^m \frac{\zeta(1+z_j+u_j)}{\zeta(1+z_j)\zeta(1+u_j)}$ $\left(\zeta(s) = \sum_1^\infty n^{-s}\right)$, G_1, G_2 are holom. in $\underbrace{D_{1/6m}^m}_{D}$ and satisfy

$V(G_1, m, D) \leq O_m(1)$, $V(G_2, m, D) \leq w(N)^{O_m(w(N))}$,

$G_1(\bar{0}, \bar{0}) = 1 + o_m(1)$, $G_2(\bar{0}, \bar{0}) = (W/\varphi(W))^m$. (Lemma 9.3)

Thus (**) can be evaluated by

<u>Lemma 9.4 (6.8 K.)</u> $R > 0$, $G = G_R(z,u)$ holom. in $\overbrace{D_\sigma^m}^{D}$ for some $\sigma > 0$, $V(G, m, D) = \exp(O_{m,\sigma}(\log^{1/3} R))$.

Then $$\frac{1}{(2\pi i)^m} \int_{\Gamma_1} \cdots \int_{\Gamma_1} G(z,u) \prod_{j=1}^m \frac{\zeta(1+z_j+u_j)}{\zeta(1+z_j)\zeta(1+u_j)} \frac{R^{z_j+u_j}}{z_j^2 u_j^2} dz_j du_j$$

$$= G(\bar{0},\bar{0}) \log^m R + \sum_{j=1}^m O_{m,\sigma}\left(V(G, j, D) \cdot \log^{m-j} R\right) + O_{m,\sigma}\left(e^{-\delta\sqrt{\log R}}\right)$$

for some $\delta = \delta(m) > 0$.

Assymptotics of (*) = (**) follows by setting $G = G_1 G_2$ and $\sigma = 1/6m$.

Lemma 9.4 ~~takes more also~~ is also used in the proof of Prop. 8.6. The proof of Lemma 9.4 takes 4 pages in the Appendix. It uses contour deformation, residues and 0-free region for $\zeta(s)$.

Wants even more details? Read and ponder the article!

On May 26, 2004, R.F. Arenstorf posted in Arxiv the preprint "There are infinitely many prime twins"... (i.e., $\exists\infty$ many $p_1 < p_2$, $p_2 - p_1 = 2$)

Last news: the first 2 primes in an A.P. appear before the number

$$2^{2^{2^{2^{2^{2^{2^{100}}}}}}}$$

(June 2, home page of T. Tao).