Lecture 9, November 28, 2019

Series of functions. Power series

Approximation by broken lines and polynomials. Recall that a function $f\colon [a,b] \to \mathbb{R}$, where $a < b$ are real numbers, is a *broken line* if f is continuous and there is a partition $a = a_0 < a_1 < \cdots < a_k = b$ of the interval $[a,b]$ such that every restriction $f \mid [a_{i-1}, a_i]$, $i = 1, 2 \ldots, k$, is a linear function. Recall that for functions $f, f_n\colon M \to \mathbb{R}$, where $n \in \mathbb{N}$ and M is a set, the notation $\lim f_n = f$ means that

$$\|f - f_n\|_\infty = \sup(\{|f(x) - f_n(x)| \mid x \in M\}) \to 0 \ \text{ for } \ n \to \infty\ .$$

The symbol $\mathrm{C}[a,b]$ denotes the set of real functions that are defined and continuous on the interval $[a,b]$. In the prof of lemma 3 in lecture 7 we proved the following result.

Proposition (approximation by broken lines). *The set of broken lines is dense in* $\mathrm{C}[a,b]$ *—for every function* $f \in \mathrm{C}[a,b]$ *there is a sequence* $(f_n) \subset \mathrm{C}[a,b]$ *of broken lines such that* $\lim f_n = f$*.*

The disadvantage of broken lines is that they are not everywhere differentiable. The next important theorem, for the proof of which we do not have time, shows that every continuous function can be approximated to any precision by functions that have derivatives of all orders.

Theorem (the Weierstrass thm.: approximation by polynomials). *The set of polynomials is dense in* $\mathrm{C}[a,b]$ *—for every function* $f \in \mathrm{C}[a,b]$ *there is a sequence* $(f_n) \subset \mathrm{C}[a,b]$ *of polynomials such that* $\lim f_n = f$*.*

The theorem bears the name of the German mathematician *Karl Weierstrass (1815–1897).* Theory of approximations of functions is a large and interesting discipline of mathematical analysis, from which we had time to mention just the two previous results.

Primitives to continuous functions.[1] A nice application of the previous proposition and of the last theorem of the previous lecture (exchange of a

[1] I did not mention this application in the lecture.

limit and derivative) is the next theorem. A better known proof of it uses theory of Riemann integration.

Theorem (existence of antiderivatives). *Every function $f \in \mathrm{C}[a,b]$ has on (a,b) a primitive function. This is a function $g\colon (a,b) \to \mathbb{R}$ such that*

$$g' = f \quad \text{on} \quad (a,b)$$

(for simplicity we do not consider one-sided derivatives in the endpoints a and b).

Proof. Let $f \in \mathrm{C}[a,b]$. By the previous proposition we have broken lines $(f_n) \subset \mathrm{C}[a,b]$ such that $\lim f_n = f$. It is not hard to see (and without the $(R)\int$) that every broken line $h \in \mathrm{C}[a,b]$ has a (unique) primitive function $g \in \mathrm{C}[a,b]$ on (a,b) such that $g(\frac{a+b}{2}) = 0$ (Exercise 2). For instance, the function $h(x) = 1$ for $0 \le x \le \frac{1}{2}$ and $h(x) = \frac{3}{2} - x$ for $\frac{1}{2} \le x \le 1$ has the primitive $g(x) = x - \frac{1}{2}$ for $0 \le x \le \frac{1}{2}$ and $g(x) = \frac{3}{2}x - \frac{x^2}{2} - \frac{5}{8}$ for $\frac{1}{2} \le x \le 1$.

For every $n \in \mathbb{N}$ we take such primitive function g_n to f_n. By the last theorem of the previous lecture,

$$g_n \overset{\text{loc}}{\rightrightarrows} g \ \ (\text{on } (a,b))$$

for a function $g\colon (a,b) \to \mathbb{R}$ such that $g' = f$ on (a,b). □

Series of functions. Let $M \subset \mathbb{R}$ be a nonempty set and $f, f_n\colon M \to \mathbb{R}$ for $n = 1, 2, \dots$ be real functions defined on it. Notation

$$\sum f_n = \sum_{n=1}^{\infty} f_n \to f \ \ (\text{on } M)$$

means that $f_1 + f_2 + \cdots + f_n \to f$ (on M). Similarly for uniform convergence, and locally uniform convergence (for a metric space M). This way one generalizes numeric series to parametric systems of series (as we spoke about it in lecture 1). One easily generalizes the (uniform) Bolzano–Cauchy condition: $\sum_{n=1}^{\infty} f_n \rightrightarrows f$ on M for a function f if and only if

$$\forall\, \varepsilon > 0\ \exists\, n_0 :\ m \ge n \ge n_0,\ x \in M \Rightarrow |f_n(x) + f_{n+1}(x) + \cdots + f_m(x)| < \varepsilon$$

(Exercise 1). Thus it suffices to write just $\sum_{n=1}^{\infty} f_n \rightrightarrows$ (on M).

The following three theorems on series of functions directly follow from the corresponding theorems on sequences of functions. We do not give proofs for them either.

Theorem ($\sum \leftrightarrow \lim_{x\to x_0}$). *If $x_0 \in \mathbb{R}^*$, $\delta > 0$, $f_n\colon P(x_0,\delta) \to \mathbb{R}$ for $n = 1, 2, \dots$, $\sum f_n \rightrightarrows$ on $P(x_0,\delta)$, and for every n there is a finite $\lim_{x\to x_0} f_n(x)$, then the next sum and limit are defined and have equal finite values:*

$$\sum_{n=1}^{\infty}\left(\lim_{x\to x_0} f_n(x)\right) = \lim_{x\to x_0}\left(\sum_{n=1}^{\infty} f_n(x)\right)\ .$$

Theorem ($\sum \leftrightarrow \int$).[2] *If $a < b$ are real numbers, $f_n \in \mathcal{R}[a,b]$ for $n = 1, 2, \dots$, and $\sum_{n=1}^{\infty} f_n \rightrightarrows$ on $[a,b]$, then $\sum_{n=1}^{\infty} f_n \in \mathcal{R}[a,b]$ and*

$$\sum_{n=1}^{\infty}\int_a^b f_n = \int_a^b\left(\sum_{n=1}^{\infty} f_n\right)\ .$$

Theorem ($\sum \leftrightarrow \frac{d}{dx}$). *If $a < b$ are real numbers, $g, f_n\colon (a,b) \to \mathbb{R}$ for $n = 1, 2, \dots$ are functions, on (a,b) there exist derivatives f_n', $\sum_{n=1}^{\infty} f_n' \rightrightarrows g$ on (a,b), and there is an $x_0 \in (a,b)$ such that the numeric series $\sum_{n=1}^{\infty} f_n(x_0)$ converges, then $\sum_{n=1}^{\infty} f_n \rightrightarrows f$ on (a,b) for a function $f\colon (a,b) \to \mathbb{R}$ such that $f' = g$ on (a,b). Thus, under these assumptions,*

$$\left(\sum_{n=1}^{\infty} f_n\right)' = \sum_{n=1}^{\infty} f_n'\ .$$

Convergence criteria for series of functions. Most often one uses the first of the following two criteria.

Theorem (the Weierstrass test). *Let f_n for $n = 1, 2, \dots$ be real functions. If f_n are defined on a set M and the nonnegative numeric series (possibly with $+\infty$ summands)*

$$\sum_{n=1}^{\infty} M_n := \sum_{n=1}^{\infty}\|f_n\| = \sum_{n=1}^{\infty}\sup(\{|f_n(x)| \mid x \in M\})$$

[2] Better theorems on exchange of a sum and a Riemann integral are known. I will mention them here later.

converges, then $\sum_{n=1}^{\infty} f_n \rightrightarrows$ *on* M*.*

Proof. We check that $\sum_{n=1}^{\infty} f_n$ satisfies the uniform B.–C. condition: for every $x \in M$ and $m \ge n \ge 1$ the triangle inequality implies that

$$\begin{aligned} |f_n(x) + f_{n+1}(x) + \cdots + f_m(x)| &\le |f_n(x)| + |f_{n+1}(x)| + \cdots + |f_m(x)| \\ &\le M_n + M_{n+1} + \cdots + M_m \ . \end{aligned}$$

By the assumption for every given $\varepsilon > 0$ there is an n_0 such that if $m \ge n \ge n_0$ then the last sum is smaller than ε (the Cauchy condition for numeric series). The uniform B.–C. condition for the given series of functions is therefore satisfied. □

Theorem (the Dini criterion). *If the functions* f_n *are defined, continuous and nonnegative on a compact metric space* M *and their pointwise sum is continuous too, then* $\sum_{n=1}^{\infty} f_n \rightrightarrows$ *on* M*.*

Proof. This is an immediate corollary of the Dini theorem in the last lecture (Exercise 11). □

How to define a function $f\colon \mathbb{R} \to \mathbb{R}$ invariant to differentiation, that is, satisfying

$$f' = f \ \text{ on } \ \mathbb{R} \ ?$$

And does such a function exist at all? (Suppose we forgot a lot from *Mathematical Analysis I.*) As $(\frac{x^n}{n!})' = \frac{x^{n-1}}{(n-1)!}$ for $n \ge 1$ and $(\frac{x^0}{0!})' = 0$, formal exchange of summation and differentiation gives

$$\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)' = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{m=0}^{\infty} \frac{x^m}{m!} \quad (m = n-1) \ .$$

So if the function $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ is correctly defined and the exchange of summation and differentiation is permissible, this f has the required property that $f' = f$. Both conditions are satisfied: by the Weierstrass test $\sum_{n=0}^{\infty} \frac{x^n}{n!} \rightrightarrows$ on every interval $(-R, R)$ with $R > 0$ (Exercise 3), so $f(x)$ is indeed correctly defined, and by the earlier theorem the exchange of summation and differentiation is permissible. We have proven that the exponential function $\exp(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ does not change when differentiated.

Power series. For real numbers x_0 and $a_0, a_1, \ldots$ we define the series of functions

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \ .$$

This is a *power series centered at* x_0 *and with coefficients* a_n, $n \in \mathbb{N}_0$. It always converges at its center and $f(x_0) = a_0$, but it may happen that it converges for no other $x \neq x_0$ (Exercise 4). In sequel we restrict for simplicity to power series centered at zero.

Theorem (Hadamard's on the convergence radius). *Let* $\sum_{n \geq 0} a_n x^n$ *be a power series centered at* 0 *and the quantity* $R \in [0, +\infty) \cup \{+\infty\}$ *be defined by the formula*

$$R = \frac{1}{\limsup_{n \to \infty} |a_n|^{1/n}} \ ,$$

where we set $\frac{1}{0} = +\infty$ *and* $\frac{1}{+\infty} = 0$. *Then for every* $x \in \mathbb{R}$ *we have*

$$\begin{aligned} |x| < R \ &\Rightarrow \ \sum_{n \geq 0} a_n x^n \ \text{ absolutely converges and} \\ |x| > R \ &\Rightarrow \ \sum_{n \geq 0} a_n x^n \ \text{ diverges} \ . \end{aligned}$$

We call R *the* convergence radius (of the power series $\sum_{n \geq 0} a_n x^n$), *and the interval* $(-R, R)$ *the* convergence interval.

Proof. If $0 < R < +\infty$ and $x \in \mathbb{R}$ then

$$\limsup_{n \to \infty} |a_n x^n|^{1/n} = |x| \limsup_{n \to \infty} |a_n|^{1/n} = \frac{|x|}{R} \ .$$

By the Cauchy root test (see *MA I*) the series $\sum_{n \geq 0} a_n x^n$ absolutely converges for $|x| < R$, and diverges for $|x| > R$. For $R = +\infty$ one has $\limsup_{n \to \infty} |a_n|^{1/n} = 0$, thus the last equality in the computation turns in $= 0$. By the Cauchy root test our series absolutely converges for every $x \in \mathbb{R}$. For $R = 0$ one has $\limsup_{n \to \infty} |a_n|^{1/n} = +\infty$, thus for every nonzero $x \in \mathbb{R}$ the last equality in the computation turns in $= +\infty$. Again by the Cauchy root test our series diverges for every nonzero $x \in \mathbb{R}$. $\square$

This is one of the best known results on power series, due to the French mathematician *Jacques Hadamard (1865–1963).* R is called convergence radius because for power series in complex domain (with $a_n, x \in \mathbb{C}$) R is the radius of the closed disc centered at the origin, inside which the power series absolutely converges, and outside of which it diverges. What happens for $x = \pm R$, or in the complex domain on the circle $|x| = R$, the theorem does not say and it must be determined separately.

Clearly, $\sum_{n\geq 0} n! x^n$ has convergence radius $R = 0$ (Exercise 4), $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ has $R = +\infty$, and all three power series

$$1 + x + x^2 + x^3 + \ldots,\ 1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \ldots \ \text{and}\ 1 + x + 2x^2 + 3x^3 + \ldots$$

have $R = 1$, because of the limit $\lim n^{1/n} = 1$ (Exercise 5). The first and third power series diverge for both values $x = \pm 1$, but the second one diverges only for $x = -1$ and (non-absolutely) converges for $x = 1$.

Proposition (on $\overset{\text{loc}}{\rightrightarrows}$ of power series). *If a power series* $\sum_{n\geq 0} a_n x^n$ *has convergence radius* $R > 0$ *(possibly* $+\infty$*) then*

$$\sum_{n=0}^{\infty} a_n x^n \overset{\text{loc}}{\rightrightarrows} \quad (\textit{on } (-R, R))\ .$$

Proof. If $S \in (0, R)$ and $x \in [-S, S]$, by the Cauchy root test the series

$$\sum_{n=0}^{\infty} \|a_n x^n\| = \sum_{n=0}^{\infty} |a_n| S^n$$

converges (because $\limsup_{n\to\infty} |a_n S^n|^{1/n} = \frac{S}{R} < 1$). Hence by the Weierstrass test $\sum_{n\geq 0} a_n x^n \rightrightarrows$ on $[-S, S]$, which is equivalent to the locally uniform convergence on $(-R, R)$. □

Corollary (differentiation and integration of power series). *Suppose that a power series* $f(x) = \sum_{n\geq 0} a_n x^n$ *has convergence radius* $R > 0$ *(possibly* $+\infty$*). Then the power series*

$$g(x) = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1} \quad \textit{and} \quad h(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

have the same convergence radius R, and on the interval $(-R, R)$ their sums satisfy that

$$g' = f \quad \textit{and} \quad f' = h\ .$$

Proof. Equality of the convergence radii of $g(x)$ and $h(x)$ to R follows from the Hadamard formula and from $\lim n^{1/n} = 1$. The equalities $g' = f$ and $f' = h$ follow from the previous proposition and from the theorem on the exchange of summation and differentiation. $\square$

Thus a function given as a sum of power series has derivatives of all orders (and primitives of all orders). The function $f\colon \mathbb{R} \to \mathbb{R}$ given as $f(x) = 0$ for $x \le 0$ and $f(x) = x^2$ for $x \ge 0$ does not equal to the sum of a power series on any interval $(-\delta, \delta)$, $\delta > 0$, because $f''(0)$ does not exist. Functions expressed by sums of power series resemble polynomials, but only to some extent (Exercise 8). Sums and products of two power series are treated in Exercises 9 and 10.

Exercises

1. Prove that uniform convergence of a series of functions is equivalent to satisfaction of the uniform B.–C. condition.

2. Prove that every broken line $f\colon [a, b] \to \mathbb{R}$ has a primitive on (a, b), even with arbitrarily prescribed value $g(c) = d$ for any c in (a, b). Give a direct proof without using the Riemann integral.

3. Does $\sum_{n=0}^{\infty} \frac{x^n}{n!} \rightrightarrows$ on $\mathbb{R}$?

4. Prove that the power series $\sum_{n\ge 0} n!x^n$ converges for no $x \ne 0$.

5. Let $p \in \mathbb{R}[x]$ be arbitrary polynomial. What is the convergence radius of the power series $\sum_{n\ge 0} p(n)x^n$?

6. Determine the convergence radii of the power series

$$\sum_{n=0}^{\infty} \frac{4x^n}{3^n - 2n + 1} \quad \text{and} \quad \sum_{n=20}^{\infty} (5^n - 200n^2 + 7n - 2019)x^{3n}\ .$$

7. Determine the convergence radii of the power series

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n \quad \text{and} \quad \sum_{n=0}^{\infty} (-1)^n x^{n^2} \,.$$

8. Yes or no: a nonzero function $f(x) = \sum_{n\geq 0} a_n x^n \colon \mathbb{R} \to \mathbb{R}$ that is given as a sum of power series has, like a nonzero polynomial, only finitely many zeros (points $a \in \mathbb{R}$ for which $f(a) = 0$).

9. If $\sum_{n\geq 0} a_n x^n$ and $\sum_{n\geq 0} b_n x^n$ are power series with positive radii of convergence, what can be said about the convergence radius of the power series
$$\sum_{n\geq 0} (a_n + b_n) x^n \,?$$

10. If $\sum_{n\geq 0} a_n x^n$ and $\sum_{n\geq 0} b_n x^n$ are power series with positive radii of convergence, what can be said about the convergence radius of the power series
$$\sum_{n\geq 0} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n \,?$$
(This is so called Cauchy's product of power series.)

11. Prove the Dini criterion of uniform convergence of series of functions.