Lecture 8, November 21, 2019

More about uniform convergence. The Moore–Osgood theorem 1 and 2. Exchange of limit and integration/differentiation (without proofs)

The uniform Bolzano–Cauchy condition. One of the basic results of the theory of limits of real sequences $(a_n) \subset \mathbb{R}$ is the equivalence

$$\exists\, a \in \mathbb{R}:\ \lim a_n = a \iff \forall\, \varepsilon > 0\ \exists\, n_0 \in \mathbb{N}:\ m,\, n \geq n_0 \Rightarrow |a_m - a_n| < \varepsilon$$

— a sequence (a_n) converges if and only if it is Cauchy. This is one of the main theorems in *Mathematical Analysis I.* We generalize it to sequences of functions.

Proposition (the uniform B.–C. condition). *Let $f_n\colon M \to \mathbb{R}$, $n \in \mathbb{N}$, be real functions defined on a set M. Then*

$$\exists\,(f\colon M \to \mathbb{R}):\ f_n \rightrightarrows f\ \ (on\ M)$$
$$\iff \forall\, \varepsilon > 0\ \exists\, n_0 \in \mathbb{N}:\ m,\, n \geq n_0,\ x \in M \Rightarrow |f_m(x) - f_n(x)| < \varepsilon\ .$$

On the left side of the equivalence one can instead of $f_n \rightrightarrows f$ (on M) write $\lim f_n = f$*, with convergence in the norm $\|\cdot\|_\infty$. The right side of the equivalence is called the* uniform Bolzano–Cauchy condition.

Proof. Implication $\Rightarrow$. If f_n converge uniformly on M to f, we take n_0 in $\mathbb{N}$ such that for every $n \geq n_0$ and every $x \in M$, $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$. Then for every $m, n \geq n_0$ and $x \in M$ we have (by the Δ inequality)

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f(x) - f_m(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\ .$$

The sequence (f_n) therefore satisfies the uniform Bolzano–Cauchy condition.

Implication $\Leftarrow$. The sequence of functions $(f_n) \subset N$ is a Cauchy sequence in the (metric or normed) space of functions $(N, \|\cdot\|)$ where $N = \{f \mid f\colon M \to \mathbb{R}\}$. By Exercise 6 in lecture 6 N is a complete space. Hence there is a function $f \in N$ such that $\lim f_n = f$. So $f_n \rightrightarrows f$ (on M). □

The notation $f_n \rightrightarrows$ (on M) and $f_n \overset{\text{loc}}{\rightrightarrows}$ (on M) therefore makes sense: the sequence $(f_n) \subset N$ satisfies on M a uniform Bolzano–Cauchy condition, or it satisfies it locally, and therefore it uniformly, or locally uniformly, converges on M to a function f.

The Dini theorem. In some situations one can deduce from the pointwise or only locally uniform convergence the uniform convergence. An example of such situation is Exercise 8 in lecture 6. Now we generalize it.

Proposition (compactness $\Rightarrow \rightrightarrows$). *If functions $f_n\colon M \to \mathbb{R}$, $n \in \mathbb{N}$, are defined on a compact metric space (M, d) and $f_n \overset{\text{loc}}{\rightrightarrows}$ (on M) then $f_n \rightrightarrows$ (on M).*

Proof. For every $a \in M$ we take a ball $B_a = B(a, r_a)$, $r_a > 0$, with $f_n \rightrightarrows$ (on B_a). These balls cover M and the compactness of M implies that for some finitely many points $a_1, \dots, a_k \in M$,

$$M = \bigcup_{i=1}^{k} B_{a_i} \ .$$

For a given $\varepsilon > 0$ let $n_i \in \mathbb{N}$ be such that if $m, n \geq n_i$ and $x \in B_{a_i}$ then $|f_m(x) - f_n(x)| < \varepsilon$. Then for every $m, n \geq \max_{1 \leq i \leq k} n_i$ and $x \in M$,

$$|f_m(x) - f_n(x)| < \varepsilon$$

as well. □

A sequence of functions $f_n\colon M \to \mathbb{R}$, $n \in \mathbb{N}$ and M is a set, is *monotone* if for every $a \in M$ one has $f_1(a) \leq f_2(a) \leq \dots$ or for every $a \in M$ one has $f_1(a) \geq f_1(a) \geq \dots$.

Theorem (Dini's). *Let $f_n \to f$ (on M) for a monotone sequence of continuous real functions f_n, $n \in \mathbb{N}$, and a continuous real function f, with all functions defined on a compact metric space (M, d). Then $f_n \rightrightarrows f$ (on M).*

Proof. For a given $\varepsilon > 0$ and $n \in \mathbb{N}$ we define sets

$$I_n = \{a \in M \mid |f_n(a) - f(a)| < \varepsilon\} \ .$$

By the continuity of f_n and f all sets I_n are open. By the pointwise convergence of f_n to f, the I_n cover M. Due to the compactness of M there exist indices $n_1, \ldots, n_k$ such that $M = \bigcup_{i=1}^{k} I_{n_i}$. Since the sequence (f_n) is monotone, $I_1 \subset I_2 \subset \ldots$. Hence $M = I_n$ for every $n \geq n_0 = \max(n_1, \ldots, n_k)$. For every $n \geq n_0$ and every $x \in M = I_n$ we thus have $|f_n(x) - f(x)| < \varepsilon$. $\square$

The theorem was discovered by the Italian mathematician *Ulisse Dini (1845–1918)* who was teaching on the universities in Pisa.

The Moore–Osgood theorem. As we know, exchange of limits may change the result:

$$\lim_{x\to 1^-} \lim_{n\to\infty} x^n = \lim_{x\to 1^-} 0 = 0, \quad \text{but} \quad \lim_{n\to\infty} \lim_{x\to 1^-} x^n = \lim_{n\to\infty} 1 = 1\ .$$

We show that with uniform convergence this cannot happen. We first state and prove the theorem for the real axis $M = \mathbb{R}$, and then give its generalization for any set M. Recall the notation for the deleted neighborhoods on the real line: if $\delta > 0$ then

$$P(x_0,\, \delta) = (x_0 - \delta,\, x_0 + \delta) \setminus \{x_0\} \ \text{ for } \ x_0 \in \mathbb{R}\ ,$$

$$P(x_0,\, \delta) = (-\infty,\, -1/\delta) \ \text{ for } \ x_0 = -\infty$$

and

$$P(x_0,\, \delta) = (1/\delta,\, +\infty) \ \text{ for } \ x_0 = +\infty\ .$$

Theorem (Moore–Osgood 1). *Let $x_0 \in \mathbb{R}^*$ (the values $x_0 = \pm\infty$ are allowed), $\delta > 0$, $f_n, f \colon P(x_0, \delta) \to \mathbb{R}$, $n \in \mathbb{N}$, $f_n \rightrightarrows f$ (on $P(x_0, \delta)$), and for every $n \in \mathbb{N}$ there exists a finite limit $\lim_{x\to x_0} f_n(x) =: a_n \in \mathbb{R}$. Then there exists a finite limit*

$$\lim_{n\to\infty} a_n = L, \quad \textit{and} \quad \lim_{x\to x_0} f(x) = L\ .$$

Thus we can exchange limits without changing the result,

$$\lim_{n\to\infty} \lim_{x\to x_0} f_n(x) = \lim_{x\to x_0} \lim_{n\to\infty} f_n(x) = L\ .$$

Proof. Since $f_n \rightrightarrows$ (on $P(x_0, \delta)$), for the given $\varepsilon > 0$ there is an index n_0 such that

$$m,\, n \geq n_0,\, x \in P(x_0, \delta) \Rightarrow |f_m(x) - f_n(x)| < \varepsilon\ .$$

For fixed m and n the limit transition $x \to x_0$ preserves the inequality or makes it an equality and

$$m,\, n \ge n_0 \Rightarrow |a_m - a_n| \le \varepsilon\ .$$

Thus $(a_n) \subset \mathbb{R}$ is a Cauchy sequence and has a finite limit $\lim_{n\to\infty} a_n = L \in \mathbb{R}$.

For every $n \in \mathbb{N}$ and every $x \in P(x_0, \delta)$ the triangle inequality gives

$$|f(x) - L| \le \underbrace{|f(x) - f_n(x)|}_{A} + \underbrace{|f_n(x) - a_n|}_{B} + \underbrace{|a_n - L|}_{C}\ .$$

Let an $\varepsilon > 0$ be given. We select large enough $n_0 \in \mathbb{N}$ such that for every $x \in P(x_0, \delta)$ and $n = n_0$ one has $A, C < \frac{\varepsilon}{3}$. We choose a $\delta_0 \in (0, \delta)$ such that for $n = n_0$ and every $x \in P(x_0, \delta_0)$ one has $B < \frac{\varepsilon}{3}$. Then

$$x \in P(x_0,\, \delta_0),\ n = n_0 \Rightarrow |f(x) - L| \le A + B + C < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\ .$$

Thus $\lim_{x\to x_0} f(x) = L$. □

The theorem is called after the American mathematicians *Eliakim Hastings Moore (1862–1932)* and *William Fogg Osgood (1864–1943)*. With the help of it we can again prove that the uniform limit of continuous functions is a continuous function. Let f_n, $n \in \mathbb{N}$, and f be real functions defined on a neighborhood U of a point $a \in \mathbb{R}$, let the functions f_n be continuous at a, and $f_n \rightrightarrows f$ (on U). Then

$$\lim_{x\to a} f(x) = \lim_{x\to a} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \lim_{x\to a} f_n(x) = \lim_{n\to\infty} f_n(a) = f(a)$$

and f is continuous at a (Exercise 1).

How to generalize the Moore–Osgood theorem to functions defined on any set?[1] For a nonempty set M, any sequence

$$\mathcal{X} = (X_n) \subset \mathcal{P}(M) \ \text{ with } X_1 \supset X_2 \supset \dots$$

of its nested nonempty subsets (thus $\emptyset \ne X_n \subset M$) is a *centered system (on M)*. We say that a *function $f\colon X_1 \to \mathbb{R}$ has a limit $L \in \mathbb{R}$ along $\mathcal{X}$*, written $\lim_{\mathcal{X}} f = L$, if

$$\forall\, \varepsilon > 0\ \exists\, n \in \mathbb{N}:\ x \in X_n \Rightarrow |f(x) - L| < \varepsilon\ .$$

[1] I did not mention this generalization in the lecture.

This is the same as $\forall \varepsilon > 0 \; \exists n_0 : \; n \geq n_0, \, x \in X_n \Rightarrow |f(x) - L| < \varepsilon$— Exercise 2.

Theorem (Moore–Osgood 2). *Let $M \neq \emptyset$ be a set, $\mathcal{X} = (X_n) \subset \mathcal{P}(M)$ be a centered system on M, $f_n, f\colon X_1 \to \mathbb{R}$, $n \in \mathbb{N}$, be functions with $f_n \rightrightarrows f$ (on X_1), and for every $n \in \mathbb{N}$ let there be a finite limit $\lim_{\mathcal{X}} f_n =: a_n \in \mathbb{R}$. Then there exists a finite limit*

$$\lim_{n\to\infty} a_n = L, \quad \textit{and} \quad \lim_{\mathcal{X}} f = L \; .$$

Thus we can exchange limits without changing the result,

$$\lim_{n\to\infty} \lim_{\mathcal{X}} f_n = \lim_{\mathcal{X}} \lim_{n\to\infty} f_n = L \; .$$

Proof. Since $f_n \rightrightarrows$ (on X_1), for a given $\varepsilon > 0$ there exists an index n_0 such that

$$m, \, n \geq n_0, \, x \in X_1 \Rightarrow |f_m(x) - f_n(x)| < \varepsilon \; .$$

Let $m, n \geq n_0$ be fixed. By the definition of limit along $\mathcal{X}$ there exist indices $k_1, k_2 \in \mathbb{N}$ such that $x \in X_{k_1} \Rightarrow |f_m(x) - a_m| < \varepsilon$ and $x \in X_{k_2} \Rightarrow |f_n(x) - a_n| < \varepsilon$. For $k_0 = \max(k_1, k_2)$ we have by the triangle inequality that

$$\begin{aligned} x \in X_{k_0} \Rightarrow |a_m - a_n| &\leq |a_m - f_m(x)| + |f_m(x) - f_n(x)| + |f_n(x) - a_n| \\ &< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon \; . \end{aligned}$$

The sequence $(a_n) \subset \mathbb{R}$ is again Cauchy and $\lim_{n\to\infty} a_n = L \in \mathbb{R}$.

For every $n \in \mathbb{N}$ and every $x \in X_1$ the triangle inequality gives

$$|f(x) - L| \leq \underbrace{|f(x) - f_n(x)|}_{A} + \underbrace{|f_n(x) - a_n|}_{B} + \underbrace{|a_n - L|}_{C} \; .$$

Let an $\varepsilon > 0$ be given. We take large enough $n_0 \in \mathbb{N}$ such that for every $x \in X_1$ and $n = n_0$, $A, C < \frac{\varepsilon}{3}$ (recall that $f_n \rightrightarrows f$ (on X_1) and $a_n \to L$ for $n \to \infty$). We select a $k \in \mathbb{N}$ such that for $n = n_0$ and every $x \in X_k$, $B < \frac{\varepsilon}{3}$. Then

$$x \in X_k, \; n = n_0 \Rightarrow |f(x) - L| \leq A + B + C < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \; .$$

Hence $\lim_{\mathcal{X}} f = L$. $\square$

This theorem generalizes its first version (Exercise 5).

The exchange of limit and integration/differentiation. Because of lack of time we will not prove the corresponding theorems.

Theorem (exchange of $\lim_{n\to\infty}$ and $\int$).[2] *Let $f_n, f\colon [a,b] \to \mathbb{R}$, where $a < b$ are real numbers and $n \in \mathbb{N}$, are functions, f_n are Riemann-integrable on $[a,b]$, and $f_n \rightrightarrows f$ (on $[a,b]$). Then f is Riemann-integrable on $[a,b]$ too and*

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f \ .$$

Theorem (exchange of $\lim_{n\to\infty}$ and $\frac{d}{dx}$). *Let $-\infty \le a < b \le +\infty$ with $a, b \in \mathbb{R}^*$ and $f_n\colon (a,b) \to \mathbb{R}$, $n \in \mathbb{N}$, be such functions that (i) every f_n has on (a,b) derivative f_n', (ii) $f_n' \overset{\text{loc}}{\rightrightarrows} g$ (on (a,b)) for a function $g\colon (a,b) \to \mathbb{R}$, and (iii) there is a point $c \in (a,b)$ such that the sequence $(f_n(c)) \subset \mathbb{R}$ converges. Then*

$$f_n \overset{\text{loc}}{\rightrightarrows} f \quad (\text{on } (a,b))$$

for a function $f\colon (a,b) \to \mathbb{R}$ such that $f' = g$ on (a,b).

Integration improves functions, discontinuous ones become continuous, but differentiation spoils them, derivative of a continuous function may be discontinuous. Thus the hypotheses of the last theorem have to involve the sequence of derivatives (f_n') rather than (f_n).

We give three examples illustrating necessity of hypotheses of the last theorem. The functions

$$f_n(x) = \frac{\sin(nx)}{n} \rightrightarrows 0$$

on $\mathbb{R}$, but the sequence of derivatives $(\cos(nx))$ does not converge pointwisely for many numbers $x \in \mathbb{R}$ (Exercise 3). Thus the uniform convergence of (f_n) does not imply the convergence of (f_n'). The functions

$$f_n(x) = \sqrt{x^2 + \frac{1}{n^2}} \rightrightarrows |x|$$

[2] Better theorems on exchange of a limit and (Riemann) integration are known. I will mention them here later.

on $\mathbb{R}$ (Exercise 4) and have derivatives $f_n'(x)$ on $\mathbb{R}$, but the limit function $f(x) = |x|$ does not have derivative at 0. The uniform convergence (f_n) to f therefore does not ensure the existence of f'. Finally, the functions $f_n(x) = n$ have derivatives $(f_n') = (0)$ clearly converging on $\mathbb{R}$ uniformly to the zero function, but the original sequence (f_n) does not converge even pointwisely. Thus assumptions (i) and (ii) are met, but not (iii), and the conclusion of the theorem does not hold.

Exercises

1. Explain each equality in the computation proving (by means of the Moore–Osgood theorem 1) continuity of the uniform limit at a point a.

2. Prove equivalence of the two definitions of the limit along $\mathcal{X}$.

3. Find some $x \in \mathbb{R}$ such that $(\cos(nx))$ does not converge.

4. Show that $\sqrt{x^2 + \frac{1}{n^2}} \rightrightarrows |x|$ (on $\mathbb{R}$).

5. Explain why the Moore–Osgood theorem 2 generalizes the Moore–Osgood theorem 1.

6. Is it true that
$$\lim_{n\to\infty} \int_0^1 f_n = \int_0^1 \lim_{n\to\infty} f_n \ ,$$
if $f_n(x) = nx(1-x)^n$?

7. Compute the limit
$$\lim_{n\to\infty} \int_0^{\pi/2} (\sin^{n+1} x - \sin^n x)\ dx$$
and justify your computation.

8. Compute the limit
$$\lim_{n\to\infty} \int_0^1 (1 + x/n)^n\ dx$$
and justify your computation.

9. This was probably mentioned before in a particular case but we still give the general version. Let $f_n \to f$ on M but $f_n \not\rightrightarrows f$ on M. Prove that there does not exist an inclusion-maximal set $A \subset M$ such that $f_n \rightrightarrows f$ on A.