Lecture 7, November 14, 2019

Proof of existence of a continuous but non-differentiable function

We prove the following

Theorem (continuous but non-differentiable function). *There exists a function* $f \in \mathrm{C}[0,1]$ *such that for every* $x \in [0,1]$ *and every* $\delta > 0$,

$$\sup\left(\left\{\left|\frac{f(y)-f(x)}{y-x}\right| \;\middle|\; y \in P(x,\,\delta) \cap [0,\,1]\right\}\right) = +\infty\ .$$

This function is of course continuous but is not differentiable at any point of the interval $[0,1]$.

Differentiability of a function at a given point means existence of a finite derivative at the point, for the endpoints of the interval meant as one-sided, and $P(x,\delta) = (x-\delta,x) \cup (x,x+\delta)$. We prove the theorem by means of four lemmas.

Lemma 1. *A function* $f \in \mathrm{C}[0,1]$ *has the property in the theorem, if it has the property that for every* $x \in [0,1]$,

$$\sup\left(\left\{\left|\frac{f(y)-f(x)}{y-x}\right| \;\middle|\; y \in [0,\,1] \setminus \{x\}\right\}\right) = +\infty\ .$$

The parameter δ *in the theorem therefore can be omitted.*

Proof. We assume that f has for every x in $[0,1]$ this property. For every $x \in [0,1]$ and every $\delta > 0$, the set

$$Q(x,\,\delta) = [0,\,1] \setminus U(x,\,\delta) \;\; (U(x,\,\delta) = (x-\delta,\,x+\delta))$$

is compact (Exercise 1), and we denote by $M_{x,\delta}$ the maximum value of the continuous function $g(y) = |(f(y)-f(x))/(y-x)|$ on it. For every $x \in [0,1]$, every $\delta > 0$, and every $c > M_{x,\delta}$ there is by the assumption a y in $[0,1]$, $y \neq x$, such that

$$\left|\frac{f(y)-f(x)}{y-x}\right| > c\ .$$

But then $y \notin Q(x,\delta)$, hence $y \in U(x,\delta)$ and $y \in P(x,\delta)$ (since $y \neq x$), and we see that f has the property in the theorem. $\square$

Lemma 2. *Suppose that (M,d) is a metric space, $(x_n) \subset M$ is a sequence of points converging to a point $x_0 \in M$, and $f_n\colon M \to \mathbb{R}$, $n \in \mathbb{N}$, is a sequence of functions converging in the norm $\|\cdot\|_\infty$ to a continuous function $f\colon M \to \mathbb{R}$. Then*

$$\lim f_n(x_n) = f(x_0)\ .$$

Proof. By the triangle inequality,

$$|f_n(x_n) - f(x_0)| \le |f_n(x_n) - f(x_n)| + |f(x_n) - f(x_0)|\ .$$

The first $|\cdots|$ on the right side is $< \varepsilon/2$ whenever $n \ge n_0$, because $\|f - f_n\| \to 0$. The same holds for the second $|\cdots|$ whenever $n \ge n_1$, due to the continuity of f at x_0 (Heine's definition of continuity is used). Hence $|f_n(x_n) - f(x_0)| < \varepsilon$ for $n \ge \max(n_0, n_1)$. □

Heine's definition of continuity tells us that for a function f continuous at a point a, $\lim a_n = a$ implies that $\lim f(a_n) = f(a)$. The previous lemma is a certain generalization.

Broken lines. The next two lemmas, or more precisely their proofs, use broken lines. A *broken line through the points* (a_0, b_0), $(a_1, b_1), \ldots, (a_k, b_k)$ in the plane (in this order), where $a_0 < a_1 < \cdots < a_k$, is the function $f\colon [a_0, a_k] \to \mathbb{R}$ defined on every interval $[a_{i-1}, a_i]$, $i = 1, 2, \ldots, k$, by

$$f(x) = \frac{(b_i - b_{i-1})(x - a_{i-1})}{a_i - a_{i-1}} + b_{i-1}\ .$$

Its graph on $[a_{i-1}, a_i]$ is the straight segment connecting the points (a_{i-1}, b_{i-1}) and (a_i, b_i). These are the *segments of the broken line.* Every broken line is a continuous function (Exercise 9).

Slope of a line in the plane given by the equation $y = ax + b$ is the number a. Slope of a straight segment is the slope of the line extending the segment. A *secant line* of a function $f\colon M \to \mathbb{R}$, $M \subset \mathbb{R}$, is a line going through two different points of the graph of f.

Lemma 3. *For every $\varepsilon > 0$ and every $f \in \mathrm{C}[0,1]$ there is a $g \in \mathrm{C}[0,1]$ and a real $M > 0$ such that $\|f - g\| < \varepsilon$ and for every two distinct points x and y in $[0,1]$,*

$$\left|\frac{g(y) - g(x)}{y - x}\right| < M\ .$$

Thus every continuous function on $[0,1]$ *can be approximated arbitrarily tightly by a continuous function whose secant lines have bounded slopes.*

Proof. Interval $[0,1]$ is compact and therefore f is even uniformly continuous (Exercise 2). For every large enough $m \in \mathbb{N}$ and every $i = 0, 1, \ldots, m$ we thus have the implication

$$\frac{i}{m} \le x \le \frac{i+1}{m} \Rightarrow |f(i/m) - f(x)|,\ |f((i+1)/m) - f(x)| < \varepsilon/2\ .$$

We take the broken line g through the points $(i/m, f(i/m))$, $i = 0, 1, \ldots, m$. It satisfies this implication too and even for the same m (Exercise 3), hence for every x in $[0,1]$ one has $|f(x) - g(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ (Exercise 4) and g has the first property. Since for every two distinct numbers x and y in $[0,1]$ it is true that

$$\left|\frac{g(y)-g(x)}{y-x}\right| \le s\ ,$$

where s is in absolute value largest slope of a segment of g (Exercise 5), g has the second property as well. □

Lemma 4. *For every small* $\varepsilon > 0$ *and every large* $T > 0$ *there is a* $g \in \mathrm{C}[0,1]$ *such that* $\|g\| < \varepsilon$ *and for every* $x \in [0,1]$ *there is a* $y \in [0,1]$ *different from* x *such that*

$$\left|\frac{g(y)-g(x)}{y-x}\right| > T\ .$$

Thus there exist continuous and small functions, defined on $[0,1]$*, with steep secant line through every point of the graph.*

Proof. For given $\varepsilon > 0$ and $T > 0$ we select sufficiently large even $m \in \mathbb{N}$ with $2m\varepsilon/3 > T$, and consider the broken line g through the $m+1$ points in the plane

$$\big(i/m,\ (\varepsilon/3)(1-(-1)^i)\big),\ i = 0,\ 1,\ \ldots,\ m\ .$$

It starts in the point $(0,0)$, ends in $(1,0)$, and consists of $m/2$ sharp tips with height $2\varepsilon/3$ and bases of width $2/m$. So $\|g\| = 2\varepsilon/3 < \varepsilon$. Through any point u of the graph of g we lead the secant line extending the segment of g containing u (there may be two such segments, then we select any of them). This line has slope in absolute value larger than T, because both sides of every tip have in absolute value slope $(2\varepsilon/3)/(1/m) = 2m\varepsilon/3 > T$. □

Proof of the theorem. For $n \in \mathbb{N}$ we define sets

$$A_n = \{f \in \mathrm{C}[0,\, 1] \mid \exists\, x \in [0,\, 1]\; \forall\, y \in [0,\, 1] \setminus \{x\} :\; |\tfrac{f(y)-f(x)}{y-x}| \le n\}\ .$$

It suffices to prove that every A_n is a meager set in $\mathrm{C}[0,1]$: since the space $\mathrm{C}[0,1]$ is complete (by the proposition in the last lecture), Baire's theorem says that there is a function

$$f \in \mathrm{C}[0,1] \setminus \bigcup_{n=1}^{\infty} A_n\ .$$

Clearly f is a continuous function defined on $[0,1]$ that is outside every of the sets A_n. It therefore has the property in lemma 1 and thus the property in the theorem, and is not differentiable at any point of the interval $[0,1]$.

We prove that every set $A_n \subset \mathrm{C}[0,1]$ is closed and contains no ball, for every ball $B(f,r) \subset \mathrm{C}[0,1]$ one has $B(f,r) \not\subset A_n$. This implies that A_n is meager (Exercise 6). We prove closedness of A_n by closedness to limits. Let $(f_k) \subset A_n$ be a sequence of points in A_n with $\lim_{k\to\infty} f_k = f \in \mathrm{C}[0,1]$ (so $f_k \rightrightarrows f$ on $[0,1]$, we show that $f \in A_n$). As $f_k \in A_n$, there is a number $x_k \in [0,1]$ such that for every $y \in [0,1]$ different from x_k one has $|\frac{f_k(y)-f_k(x_k)}{y-x_k}| \le n$. A theorem in *Mathematical Analysis I* says that the sequence $(x_k) \subset [0,1]$ has a convergent subsequence with limit in $[0,1]$. To simplify notation we assume that already $\lim_{k\to\infty} x_k = x_0 \in [0,1]$. For every $y \in [0,1]$ different from x_0 then by the property of x_k and by lemma 2 we have

$$n \ge \lim_{k\to\infty} \left| \frac{f_k(y)-f_k(x_k)}{y-x_k} \right| = \left| \frac{f(y)-f(x_0)}{y-x_0} \right|$$

because non-sharp inequality is preserved in the limit. The number x_0 thus witnesses that $f \in A_n$ and A_n is closed.

It remains to find in a given ball $B(f,r) \subset \mathrm{C}[0,1]$ a point (i.e. a function) g outside A_n. We define g as $g = g_1 + g_2$ where we find g_i by lemmas 3 and 4. First we use lemma 3 to find a function $g_1 \in \mathrm{C}[0,1]$ and a constant $M > 0$ such that $\|f - g_1\| < r/2$ and all secant lines of g_1 have in absolute value slopes $< M$. Then we find by lemma 4 a function $g_2 \in \mathrm{C}[0,1]$ such that $\|g_2\| < r/2$ and for every point of the graph of g_2 there is a secant line of g_2 through it with slope in absolute value more than $M + n$. By the triangle inequality, $\|f - g\| \le \|f - g_1\| + \|g_2\| < r/2 + r/2 = r$ and $g \in B(f,r)$. Let

$x \in [0,1]$ be arbitrary. Using the property of g_2 we take a $y \in [0,1] \setminus \{x\}$ such that $|(g_2(y)-g_2(x))/(y-x)| > M+n$. Then

$$\begin{aligned} \left|\frac{g(y)-g(x)}{y-x}\right| &= \left|\frac{g_2(y)-g_2(x)}{y-x} + \frac{g_1(y)-g_1(x)}{y-x}\right| \\ &\geq \left|\frac{g_2(y)-g_2(x)}{y-x}\right| - \left|\frac{g_1(y)-g_1(x)}{y-x}\right| \\ &> (M+n)-M = n\ , \end{aligned}$$

so that $g \notin A_n$. On the first line we used the definition of g, on the second an inequality of Exercise 7, and on the third the properties of the functions g_1 and g_2. □

Exercises

1. Why is for $x \in \mathbb{R}$ and $\delta > 0$ the set $[0,1] \setminus U(x,\delta)$ compact?
2. Let (M,d) be a compact metric space and $f\colon M \to \mathbb{R}$ be a continuous function. Prove that f is uniformly continuous (i.e. $\forall \varepsilon > 0\ \exists \delta > 0 :$ $d(a,b) < \delta \Rightarrow |f(a)-f(b)| < \varepsilon$).
3. Let $f\colon [a,b] \to \mathbb{R}$ be a linear function. Show that for every $x \in [a,b]$ one has $\min(f(a),f(b)) \leq f(x) \leq \max(f(a),f(b))$.
4. Let $f,g\colon [0,1] \to \mathbb{R}$ be functions, $a_0 = 0 < a_1 < a_2 < \cdots < a_k = 1$ be a division of the interval $[0,1]$, $f(a_i) = g(a_i)$, for every x the function f satisfies $a_{i-1} \leq x \leq a_i \Rightarrow |f(x)-f(a_{i-1})|, |f(x)-f(a_i)| < \varepsilon$, and the same holds for the function g. Prove that then for every $x \in [0,1]$ one has $|f(x)-g(x)| < 2\varepsilon$.
5. Let $f\colon [0,1] \to \mathbb{R}$ be a broken line and $S = \max|s|$, taken over all slopes s of its segments. Show that then for the slope t of every secant line of f we have $|t| \leq S$.
6. Prove that every closed set X in a metric space with empty interior (i.e. X contains no ball) is meager.
7. Prove the inequality $|a+b| \geq |a| - |b|$, $a,b \in \mathbb{R}$.

8. Determine the subsets of the definition domains on which the following sequences of functions converge pointwisely, uniformly, and locally uniformly. What are the limit functions?

   (a) $f_n(x) = \frac{1}{x+n}$ on $\mathbb{R}$.
   (b) $f_n(x) = x^n - x^{3n}$ on $[0, 1]$.
   (c) $f_n(x) = x^{n+1} - x^{n-1}$ on $[0, 1]$.
   (d) $f_n(x) = x^n - x^{n+1}$ on $\mathbb{R}$.

9. Why is every broken line a continuous function?