Lecture 5, October 31, 2019

Baire' theorem. Perfect sets. Chapter 2. Types of convergence of sequences of functions

Baire's theorem. There are no "holes" in complete spaces and these spaces are rich on points. Banach's fixed-point theorem for example says that each equation $f(x) = x$ with a contractive selfmap f of a complete space has a solution. Baire's theorem describes richness of complete spaces from another angle: one cannot exhaust a complete space by "infrequent points". To state the theorem we need two new notions. The first one specifies the "infrequent points". A set $X \subset M$ in a general metric space (M, d) is *meager* if

$$\forall \text{ ball } B \subset M \ \exists \text{ ball } B' \subset B : \ B' \cap X = \emptyset \ .$$

The second notion is, for an $a \in M$ and a real $r > 0$, the *closed ball* $\overline{B}$ *(with center* a *and radius* r*)*,

$$\overline{B} = \overline{B}(a, r) = \{x \in M \mid d(x, a) \le r\} \ .$$

The only difference compared to ordinary balls is that $\overline{B}$ includes also points having distance to a exactly r. It is easy to see that $\overline{B}$ is a closed set (Exercise 1), $B(a,r) \subset \overline{B}(a,r)$, and that for every positive $s < r$ one has $\overline{B}(a,s) \subset B(a,r)$ (Exercise 2).

Theorem (R.-L. Baire, 1899). *Let* (M, d) *be a complete metric space and* $X_n \subset M$, $n \in \mathbb{N}$, *be meager sets. Then*

$$M \setminus \bigcup_{n=1}^{\infty} X_n \neq \emptyset \ .$$

A complete metric space therefore never equals to an at most countable union of meager sets (Exercise 3).

Proof. We define a sequence of closed balls

$$\overline{B}_1 \supset \overline{B}_2 \supset \overline{B}_3 \supset \ldots, \overline{B}_n = \overline{B}(a_n, r_n) \ ,$$

such that (i) $\overline{B}_n \cap X_n = \emptyset$ and (ii) $\lim r_n = 0$. It yields a point $a \in M$ that lies in none of the sets X_n. Namely, the sequence of centers $(a_n) \subset M$ is Cauchy ($m, n \ge n_0 \Rightarrow a_m, a_n \in \overline{B}_{n_0}$, thus $d(a_m, a_n) \le 2r_{n_0}$) and its limit

$a = \lim a_n \in M$ has to lie in each $\overline{B}_n$ ($n \ge n_0 \Rightarrow a_n \in \overline{B}_{n_0}$, so $\lim a_n \in \overline{B}_{n_0}$ by Exercise 1). Hence $a \notin X_n$ for every $n \in \mathbb{N}$.

We define such sequence of closed balls. Let $B_1 \subset M$ be any ball and $B_1' \subset B_1$, $B_1' = B(a_1, r_1')$, be a subball disjoint to X_1. We set $\overline{B_1} = \overline{B}(a_1, r_1)$ where $r_1 = r_1'/2$. Then (by Exercise 2) $\overline{B_1} \subset B_1'$ and therefore $\overline{B_1}$ is disjoint to X_1. Let us suppose that for $n \in \mathbb{N}$ and $i = 1, 2, \dots, n$ we have already defined the closed balls

$$\overline{B}_1 \supset \overline{B}_2 \supset \cdots \supset \overline{B}_n,\ \overline{B}_i = \overline{B}(a_i,\, r_i)\ ,$$

such that $\overline{B}_i \cap X_i = \emptyset$ and (for $i < n$) $r_{i+1} \le \frac{r_i}{2}$. We define $\overline{B}_{n+1}$ as follows. Since the set X_{n+1} is meager, there is a ball $B_{n+1}' = B(a_{n+1}, r_{n+1}') \subset B(a_n, r_n) \subset \overline{B}_n$ disjoint to X_{n+1}. We set

$$\overline{B}_{n+1} = \overline{B}(a_{n+1},\, r_{n+1}),\ r_{n+1} = \min(r_{n+1}'/2,\, r_n/2)\ .$$

It is easy to see that $\overline{B}_{n+1}$ is contained in B_{n+1}', thus in $\overline{B}_n$, and is therefore disjoint to X_{n+1}. Also, $r_{n+1} \le r_n/2$. It is clear that the sequence of nested closed balls $(\overline{B}_n)$ defined in this way has properties (i) and (ii). □

The most famous application of Baire's theorem is the proof of existence of a continuous function $f\colon [0,1] \to \mathbb{R}$ that does not have finite derivative $f'(a)$ in any point $a \in [0,1]$ of the interval (for $a = 0$ or 1 we mean one-sided derivative). Maybe this will be told in the second chapter. We conclude the first chapter with a simpler application.

Corollary (on perfect sets). *Every nonempty and at most countable closed sets $X \subset \mathbb{R}$ has an isolated point (Exercise 4). In fact, this holds in any complete metric space (M, d).*

Proof. Suppose the subset $X \subset \mathbb{R}$ is nonempty, closed, and has no isolated point. The forthcoming argument works in any complete space but for some reason we prefer to have $\mathbb{R}$ in mind. We show that X is uncountable. By Exercise 15 in the last lecture the Euclidean subspace X is complete. Thus it suffices to show that for each $a \in X$ the singleton set $\{a\}$ is meager (in X). By the Baire theorem, the union

$$X = \bigcup_{a \in X} \{a\}$$

then cannot be countable. So let $a \in X$ and $B = B(b, r) \subset X$ with $b \in X$ be any ball in the space X. If $b \neq a$ then every subball $B' \subset B$, $B' = B(b, s)$ with $s = \min(r, d(b, a) = |b - a|)$, is disjoint to $\{a\}$ (i.e. $a \notin B'$). Let $b = a$. Since a is not an isolated point of the set X, there is a point $c \in X \cap B(a, r)$, $c \neq a$. We set $B' = B(c, s)$ with $s = \min(d(c, a), r - d(c, a))$ (in $\mathbb{R}$ we have, of course, $d(c, a) = |c - a|$). It is clear that $B' \subset B$ and $a \notin B'$. Each one-element set $\{a\}$, $a \in X$, is meager in X. □

A subset of a metric space is *perfect* if it is closed and has no isolated point. The corollary thus says that in a complete metric space every nonempty perfect set is uncountable.

Chapter 2: sequences and series of functions

In *Mathematical Analysis I* we investigated limits $a = \lim_{n \to \infty} a_n$ of real sequences $(a_n) \subset \mathbb{R}$ and sums $s = \sum_{n=1}^{\infty} a_n$ of the corresponding infinite series, which are the limits $\lim_{n \to \infty} (a_1 + a_2 + \cdots + a_n)$ of sequences of partial sums. In this chapter we generalize it: instead of a single sequence $(a_n) \subset \mathbb{R}$, a parametric set $(a_n(x)) \subset \mathbb{R}$, $x \in M \subset \mathbb{R}$ (in this chapter M denotes a nonempty set of real numbers), of such sequences is given. Thus for each element $x \in M$ we have a real sequence $(a_n(x))$ or, equivalently, for each index $n \in \mathbb{N}$ we have a function $a_n \colon M \to \mathbb{R}$. We will investigate how the limit $a = a(x)$ or the sum $s = s(x)$ depend on $x \in M$. Instead of a_n we prefer to write f_n which clearly reminds of functions.

Three kinds of convergence of sequences of functions. Let $M \subset \mathbb{R}$ be a nonempty set and $f_n, f \colon M \to \mathbb{R}$, $n \in \mathbb{N}$, be functions defined on it. The functions f_n *pointwisely converge on the set M to the function f*, written

$$f_n \to f \text{ on } M \ ,$$

if for every $x \in M$ one has $\lim f_n(x) = f(x)$. In other words,

$$\forall\, \varepsilon > 0\; \forall\, x \in M\; \exists\, n_0 = n_0(x) \in \mathbb{N} :\; n \geq n_0 \Rightarrow |f_n(x) - f(x)| < \varepsilon$$

(the precise formal translation should be $\forall\, x \in M\; \forall\, \varepsilon > 0 \ldots$, but from the logical point of view the order of quantifiers of the same kind is irrelevant, and this order is better because of the next definition). The index n_0 in general depends on the selected $x \in M$, which is captured by the notation $n_0 = n_0(x)$.

In fact, we are given a function $n_0\colon M \to \mathbb{N}$ which may be unbounded when some "troublesome" points x require larger and larger values of n_0. To be completely precise we should write even $n_0 = n_0(\varepsilon, x)$ because n_0 in general depends also on ε, and thus we have actually a function $n_0\colon (0, +\infty) \times M \to \mathbb{N}$; but we take ε in this argument fixed.

If the function $n_0(x)$ is constant and a single n_0 fits all $x \in M$, we speak of *uniform convergence*. Logically this means that the second and third quantifier in the above formula are exchanged:

$$\forall\, \varepsilon > 0\ \exists\, n_0\ \forall\, x \in M :\ n \geq n_0 \Rightarrow |f_n(x) - f(x)| < \varepsilon\ .$$

We say then that the functions f_n *converge uniformly on the set M to the function f* and write

$$f_n \rightrightarrows f \text{ on } M\ .$$

From the practical perspective (in applications of uniform convergence) it is often useful to relax the constantness of the function $n_0\colon M \to \mathbb{N}$ only to a local condition. We say that the functions f_n *converge locally uniformly on the set M to the function f* and write

$$f_n \overset{\text{loc}}{\rightrightarrows} f \text{ on } M\ ,$$

if for every $x \in M$ there is a $\delta > 0$ such that $f_n \rightrightarrows f$ on $M \cap (x - \delta, x + \delta)$.

An example with powers. Let $M = [0, 1]$, $f_n(x) = x^n$ for $n \in \mathbb{N}$, and $f(x) = 0$ for $0 \leq x < 1$ and $f(1) = 1$. It is clear that $f_n \to f$ on $[0, 1]$. Note that although all functions f_n are continuous, their pointwise limit f is not continuous. It is also easy to see that this convergence is not uniform. Since for each fixed $n \in \mathbb{N}$ we have $\lim_{x \to 1^-} x^n = 1$, we can select for each $n \in \mathbb{N}$ a point $a_n \in (0, 1)$ such that $f(a_n) = a_n^n > \frac{1}{2}$ (or that $f(a_n) > c$ for any other constant $c \in [0, 1)$, for example $c = 0.99999$). But then for no $\varepsilon < \frac{1}{2}$ there is an index n such that

$$\forall\, x \in M :\ |f_n(x) - f(x)| < \varepsilon\ ,$$

because for $x = a_n \in M$ this absolute value equals $|f_n(a_n) - f(a_n)| = f_n(a_n) > \frac{1}{2}$. Thus $f_n \not\rightrightarrows f$ on $M = [0, 1]$. The well known limit

$$\lim\, (1 - n^{-1})^n = e^{-1} > 3^{-1}$$

shows that we can select the points a_n so that they approach the "troublesome" point 1 with the "speed" $\frac{1}{n}$.

Exercises

1. Prove that every closed ball is a closed set.

2. Prove that for every $a \in M$ and every positive $s < r$, $\overline{B}(a, s) \subset B(a, r)$.

3. The statement of Baire's theorem has countable union $\bigcup_{n=1}^{\infty}$. How does follow from this (purely formally) that the theorem in fact holds for finite unions too?

4. Here is an attempt to construct a countable and closed set $X \subset \mathbb{R}$ without isolated points. Let $X_0 = \{0\}$. This is a closed and at most countable set but it has an isolated point. Let $X_1 = \{1/n \mid n \in \mathbb{N}\}$. The set $X_0 \cup X_1$ is clearly countable and closed and no point in X_0 is isolated. But every point in X_1 is isolated. But we can add for every $b \in X_1$ in a similar way a sequence of points converging to b. Let X_2 be the union of these sequences over all $b \in X_1$. Then $X_0 \cup X_1 \cup X_2$ is a countable and closed set and no point in $X_0 \cup X_1$ is isolated. We can similarly remove isolation of points in X_2 by adding a countable set X_3 and so on. The resulting set $X = \bigcup_{n=0}^{\infty} X_n$ is countable and closed and none of its points is isolated. But this contradicts the Baire theorem. What is wrong?

5. Prove that the union of two meager sets is a meager set.

6. Is the fact that a set $X \subset M$ is meager a relative or an absolute property?

7. For a metric space (M, d) and $X \subset M$, we call the set X *dense (in M)* if for every ball $B \subset M$, $X \cap B \neq \emptyset$. Is it true that the complement of a meager set is a dense set?

8. Is it true that the complement of a dense set is a meager set?

9. Is the intersection of two dense sets a dense set?

10. Let $f_n(x) = x^n \colon [0, 1] \to \mathbb{R}$, $n = 1, 2, \ldots$, and let f be the pointwise limit of f_n ($f(x) = 0$ for $0 \leq x < 1$ and $f(1) = 1$). Prove that the set $S = \{X \subset [0, 1] \mid X \neq \emptyset, f_n \rightrightarrows f \text{ on } X\}$ has no maximal element with respect to inclusion (the sequence (f_n) has no largest domain of uniform convergence). What are the minimal elements?