# Lecture 4, October 24, 2019

## Homeomorphisms. Handcuffs. Connected spaces. Complete spaces

**Absoluteness of compactness.** In contrast with openness and closedness, compactness is an absolute property and does not depend on the space or subspace we consider the given set in (Exercise 1).

**Proposition (compactness and homeomorphism).** *If $f\colon M \to N$ is a continuous and injective map between metric spaces and $M$ is compact, then the inverse map $f^{-1}\colon f(M) \to M$ is continuous. The map $f$ is therefore a homeomorphism between the spaces $M$ and $f(M)$ (where the last one is given as a subspace of the space $N$).*

*Proof.* By Exercise 6 in the last lecture $f(M)$ is a compact subset of the space $N$. By the definition of a compact set in lecture 2 the space $f(M)$ is compact. Let $X \subset M$ be an arbitrary closed subset of the space $M$. By Exercise 3 in the last lecture $X$ is compact. By Exercise 6 in the last lecture $f(X)$ is a compact subset of the space $f(M)$. By Exercise 4 in the last lecture $f(X)$ is a closed subset of the space $f(M)$. But

$$\left(f^{-1}\right)^{-1}(X) = f(X)$$

— the preimage of any closed subset of the space $M$ in the map $f^{-1}$ is a closed subset of the space $f(M)$. The map $f^{-1}\colon f(M) \to M$ is therefore continuous by Exercise 2 in the last lecture. □

**Euclidean spaces $[a, b]$ and $S_1$.** At the closing of the last lecture we mentioned a continuous bijection between the Euclidean spaces $[0, 2\pi) \subset \mathbb{R}$ and

$$S_1 = \{(x,\, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \subset \mathbb{R}^2 \text{ (the unit circle)}\,,$$

with a discontinuous inverse. We promised to show that both spaces are not homeomorphic. But this is obvious (Exercise 2) because the former is not compact, but the latter is (for example by Exercise 5 in the last lecture). This leads naturally to the next question.

Are the Euclidean spaces $[a, b]$ and $S_1$ homeomorphic?

(Here $a, b \in \mathbb{R}$ with $a < b$.) Both are compact and therefore the previous argument does not apply. A student asked in the lecture how is it with a bijection between $[a, b]$ and $S_1$. A bijection between $[a, b)$ and $S_1$ is clear, it is easy to wind this interval bijectively around the circle. Last time we explicitly defined such winding for $[0, 2\pi)$, but how do we bijectively "wind" $[a, b]$? With the help of the Cantor–Bernstein theorem (Exercise 3): If there are injections $f\colon A \to B$ and $g\colon B \to A$ ($A$ and $B$ are sets) then there exists a bijection $h\colon A \to B$ (and always $h(x) = f(x)$ or $h(x) = g^{-1}(x)$). Before we resolve by means of connected spaces the problem whether $[a, b]$ and $S_1$ are homeomorphic, we consider a puzzle that is related to homeomorphisms.

**A puzzle with handcuffs.** In our three-dimensional space $\mathbb{R}^3$ with the coordinates $(x, y, z)$, three sets $P, K_1, K_2 \subset \mathbb{R}^3$ are given. The set $P$ is three-dimensional "handcuffs", two rings joined by a bar, and $K_1, K_2$ is a one-dimensional circle in two positions; $P \cap K_1 = P \cap K_2 = \emptyset$. We analytically describe and define all three sets below. In the first position $P \cup K_1$ the circle $K_1$ and each of the two rings are linked. In the second position $P \cup K_2$ the circle $K_2$ and only one ring are linked. The task is to transform the first position in the second (or the other way around), by only using continuous and injective transformations. The circle is made from a perfectly elastic wire which can be arbitrarily stretched and bent. The handcuffs is made of a perfectly deformable material, perhaps of some ideal modeling clay, and one can form it and mould arbitrarily. But only in a continuous and injective way, material cannot non-injectively penetrate through itself nor can be discontinuously broken. For example, the transformation when one ring of the handcuffs is broken, the circle slides out through the gap, and the broken ring is glued back, is not admissible. What manipulation transforms the position with the circle linked in both rings in the position where it is linked in only one ring? It seems impossible: if the circle is linked in a ring, then without breaking it or the ring or without miraculously moving the circle through the ring they cannot be unlinked? The hint is to fully use the three-dimensionality and plasticity of the handcuffs. (For a solution see for example p. 25 in `https://mskrieger.files.wordpress.com/2015/12/6305_nishiyama.pdf`)

As a small workout in the analytic geometry and metric spaces we describe the handcuffs and the circle(s) by equations. We work in the Euclidean metric space $\mathbb{R}^3 = (\mathbb{R}^3, d_2)$. For $X \subset \mathbb{R}^3$ and $\delta > 0$ we introduce the set

$$(X)_\delta = \{a \in \mathbb{R}^3 \mid \exists\, b \in X:\ d_2(a, b) \le \delta\}\ .$$

It is the three-dimensional hull of $X$ with thickness $\delta$. For $u, v, r \in \mathbb{R}$ with $r > 0$ we define the sets

$$K(u, v, r) = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0 \ \& \ (x - u)^2 + (y - v)^2 = r^2\}$$

and

$$L(u, v, r) = \{(x, y, z) \in \mathbb{R}^3 \mid y = 0 \ \& \ (x - u)^2 + (z - v)^2 = r^2\} \ .$$

The first is the horizontal circle in the plane $z = 0$ with the radius $r$ and center $(u, v, 0)$. The second is the vertical circle in the plane $y = 0$ with the radius $r$ and center $(u, 0, v)$. We set

$$P = (K(-6, 0, 4))_1 \cup (\{(x,\ y,\ z) \mid -2 \le x \le 2,\ y = z = 0\})_1 \cup (K(6, 0, 4))_1$$

and

$$K_1 = L(0,\ 0,\ 6),\ \ K_2 = L(12,\ 0,\ 6) \ .$$

It is not hard to check (Exercise 4) that $P \cup K_i$, $i = 1, 2$, are the two positions of the handcuffs and the circle.

But what is meant by the "continuous and injective transformations" that are allowed for transforming $P \cup K_1$ in $P \cup K_2$? We give more details here later.

**Connected spaces.** A subset $X \subset M$ in a metric space $(M, d)$ is *clopen*, if it is both open and closed, like the sets $\emptyset$ and $M$. A space $M$ is *connected*, if it has no nontrivial (differing from $\emptyset$ and $M$) clopen subset. A (sub)set $X \subset M$ is *connected*, if the subspace $(X, d)$ is connected. Else, if $M$ or $X$ has a nontrivial clopen subset, we speak of a *disconnected* space or a *disconnected* subset. For instance $X = \{0\} \cup \{1\} \cup (11, 14] \subset \mathbb{R}$ is a disconnected set (in the Euclidean space $\mathbb{R}$) because $\{1\}$ is one of its nontrivial clopen subsets (Exercise 5). Like compactness, connectedness or disconnectedness of $X$ is an absolute property (Exercise 6).

We give an equivalent definition of disconnectedness. In set theory or in combinatorics we mean by a *partition of a set* $A$ a set $B$ with the properties: (i) $\emptyset \notin B$, (ii) $C, D \in B$ and $C \ne D \Rightarrow C \cap D = \emptyset$, and (iii) $\bigcup B = A$. It is easy to see that the following is an equivalent definition of disconnectedness of a subset $X \subset M$:

> A subset $X \subset M$ is disconnected if it has an *open* (a *closed*) *partition* $\{Y,\ Z\}$, a partition of $X$ with set $Y, Z \subset X$ that are open (closed) in $X$.

Because of the characterization of open and closed sets in a subspace (Exercise 10 in the previous lecture) we can also say that a subset $X \subset M$ is disconnected if and only if there exist open (closed) subsets $A, B \subset M$ such that $\{A \cap X, B \cap X\}$ is a partition of $X$. Intuitively, a space is disconnected if it decomposes in two nonempty and separated parts. Connectedness means that there is no such decomposition.

**Proposition (connected sets and continuous maps).** *If $f\colon M \to N$ is a continuous map between metric spaces and $X \subset M$ is connected, then the image $f(X) \subset N$ is also connected.*

*Proof.* We show that disconnectedness of $f(X)$ implies disconnectedness of $X$. Since $f(X)$ is disconnected, there are open subsets $A, B \subset N$ such that $P = \{A \cap f(X), B \cap f(X)\}$ is a partition of $f(X)$. We claim that then

$$\{f^{-1}(A) \cap X,\ f^{-1}(B) \cap X\}$$

is an open partition of $X$. Both inverse images are open sets in $M$ (by Exercise 2 in the last lecture). The two intersections are disjoint: if both $x \in f^{-1}(A) \cap X$ and $x \in f^{-1}(B) \cap X$ then we would have $f(x) \in A \cap B \cap f(X)$, contradicting that $P$ is a partition of $f(X)$. As $A \cap f(X) \neq \emptyset$, we can take an $a \in A \cap f(X)$. Then any $x \in X$ with $f(x) = a$ satisfies $x \in f^{-1}(A)$, thus $f^{-1}(A) \cap X \neq \emptyset$. In the same way we show that $f^{-1}(B) \cap X \neq \emptyset$. Finally, if $x \in X$ is arbitrary, then $f(x)$ lies in $A \cap f(X)$ or in $B \cap f(X)$ (because $P$ is a partition of $f(X)$), hence $x$ lies in $f^{-1}(A)$ or in $f^{-1}(B)$ and the union of the two intersections is $X$. $\square$

Thus two homeomorphic metric spaces are both connected or both disconnected.

**Intervals.** Recall that $X \subset \mathbb{R}$ is an *interval* if

$$b \in X, \text{ whenever } a, b, c \in \mathbb{R} \text{ satisfy } a < b < c \text{ and } a, c \in X\ .$$

For $a, b \in \mathbb{R}$ with $a < b$, here is the list of all types of intervals, sorted from the largest one to the smallest: $\mathbb{R} = (-\infty, +\infty)$, $(-\infty, a]$, $(-\infty, a)$, $(a, +\infty)$, $[a, +\infty)$, $[a, b]$, $[a, b)$, $(a, b]$, $(a, b)$, $\{a\}$, and $\emptyset$.

**Proposition (connected sets in $\mathbb{R}$).** *An Euclidean (sub)space $X \subset \mathbb{R}$ is connected if and only if it is an interval.*

*Proof.* If $X \subset \mathbb{R}$ is not an interval then there are three real numbers $a < b < c$ such that $a, c \in X$ but $b \notin X$. It is easy to see that then

$$\{X \cap (-\infty, b), X \cap (b, +\infty)\}$$

is an open partition of $X$ which is therefore disconnected. Let us suppose that $X \subset \mathbb{R}$ is disconnected and we have its closed partition

$$P = \{X \cap A, X \cap B\}$$

where $A, B \subset \mathbb{R}$ are closed subsets. Since the two intersections are nonempty and disjoint, we can take numbers $a \in X \cap A$ and $b \in X \cap B$ and assume that $a < b$. We define the number

$$c = \sup(\{x \in [a, b] \mid x \in A\}) \in [a, b]$$

(the supremum $c$ exists because $a \in A$ and $b$ is an upper bound of the set). We distinguish two cases depending on whether $c$ lies in $X$ or not.[1] If $c \notin X$ then $a < c < b$ and this triple of points shows that $X$ is not an interval. Let $c \in X$. By the property of supremum and closedness of $A$ we have that $c \in A$. Hence $c < b$ because $c \notin B$ (as $P$ is a partition of $X$). If $(c, b) \subset X$ then also $(c, b) \subset B$ (because this interval is larger that the supremum $c$), $c$ would be a limit of a sequence of points in $B$, and we would have $c \in B$ by closedness of $B$, but as we know this is not the case. Thus there is a point $d \in (c, b)$ such that $d \notin X$. The triple $a < d < b$ shows again that $X$ is not an interval. □

For instance the unit circle $S_1$ is a connected (Euclidean) space because it is a continuous image of a connected set, the interval $[0, 2\pi)$. Using the two previous propositions and Exercise 7 we can generate many connected spaces.

**The spaces $[a, b]$ and $S_1$ are not homeomorphic.** Now we prove it. It would seem that even (dis)connectedness does not help us because both spaces are connected. However, the set $[a, b] \setminus \{c\}$ is disconnected for any deleted point $c \in (a, b)$, but $S_1 \setminus \{p\}$ is connected for any deleted point $p \in S_1$ (in a moment we justify both). If a homeomorphism $f: [a, b] \to S_1$ existed, for every point $c \in (a, b)$ the image $f^{-1}(S_1 \setminus \{f(c)\}) = [a, b] \setminus \{c\}$ of the connected set $S_1 \setminus \{f(c)\}$ by the continuous map $f^{-1}$ would be disconnected,

[1] In the lecture I inadvertently skipped this point.

contrary to the above proposition. Hence there is no homeomorphism between $[a,b]$ and $S_1$. The space $[a,b]\setminus\{c\}$, $a<c<b$, is disconnected by the last proposition because it is not an interval. The space $S_1\setminus\{p\}$ is connected because it is homeomorphic to the connected space $\mathbb{R}$: if $p=(0,1)$ is the north pole, the bijection

$$f\colon S_1\setminus\{(0,\,1)\}\to X=\text{the } x\text{-axis},\;\; f(q)=\text{the intersection of } \ell \text{ and } X\;,$$

where $\ell$ is the line going through the points $(0,1)$ and $q$, is a homeomorphism of both spaces. It is easy to adapt this to other points $p\in S_1$. Connectedness and homeomorphisms are further treated in Exercises 8–11.

**Complete spaces.** A metric space $(M,d)$ is *complete* if every Cauchy sequence $(a_n)\subset M$ of points in it has a limit $\lim a_n=a\in M$. Let us recall and in fact define that a sequence $(a_n)\subset M$ is *Cauchy* if

$$\forall\,\varepsilon>0\;\exists\,n_0:\;\; m,n>n_0\Rightarrow d(a_m,\,a_n)<\varepsilon\;.$$

A subset $X\subset M$ is *complete* if the subspace $(X,d)$ is complete. The basic example of a complete space is of course the Euclidean space $\mathbb{R}=(\mathbb{R},|x-y|)$, as it is well known from the *Mathematical Analysis I* from the theorem on convergence of Cauchy sequences.

Completeness of a subset is again an absolute property (Exercise 12). Further simple properties of complete spaces are given as exercises. A complete subset is always closed (Exercise 13). Any compact subset is complete (Exercise 14). Any closed subset of a complete space is complete (Exercise 15). We present two remarkable theorems on complete spaces. We leave the proof of the first one as Exercise 16 and will give the second one and its proof in the next lecture.

**Banach's fixed-point theorem.** A selfmap $f\colon M\to M$ of a metric space $(M,d)$ is *contractive* if there is a real constant $c\in[0,1)$ (less than 1!) such that

$$\forall\,x,\,y\in M:\;\; d(f(x),\,f(y))\le c\cdot d(x,\,y)$$

— $f$ contracts distances in the ratio at least $c:1$.

**Theorem (S. Banach, 1922).** *Let* $(M,d)$ *be a complete metric space and*

$$f\colon M\to M$$

*be a contractive map. Then there exists exactly one point $x_0 \in M$ such that $f(x_0) = x_0$ ($x_0$ is a "fixed point" of the map $f$). For every point $a \in M$ the sequence*

$$(a,\ f(a),\ f(f(a)),\ f(f(f(a))),\ \dots) \subset M$$

*of iterates of the map $f$ starting at $a$ converges to $x_0$.*

For hints for a proof see Exercise 16. Complete spaces are equally important as compact spaces (connected spaces are without doubt in importance behind them) because in complete spaces we have solutions to all kinds of equations. For example, we saw in *Mathematical Analysis I* that $x^2 = 2$ has a solution in $\mathbb{R}$. Banach's fixed-point theorem guarantees solvability of a wide class of differential (and other) equations. Maybe we will hear more about it in the next chapter. Further properties of complete spaces are given in Exercises 17–20.

## Exercises

1. Let $A \subset X \subset M$ be subsets in a metric space $(M, d)$. Prove that $A$ is compact in $M$ if and only if it is compact in the subspace $X$.

2. Prove that two homeomorphic metric spaces are both compact or both non-compact.

3. How does the existence of a bijection between $[a, b]$ ($a < b$ are real numbers) and $S_1$ follow from the Cantor–Bernstein theorem?

4. Check that these analytic definitions produce as sets $P \cup K_1$ and $P \cup K_2$, the handcuffs and the circle in the two described positions.

5. How many clopen sets are there in the space $X = \{0\} \cup \{1\} \cup (11, 14] \subset \mathbb{R}$?

6. Let $A \subset X \subset M$ be subsets in a metric space $(M, d)$. Prove that $A$ is connected in $M$ iff $A$ is connected in the subspace $X$.

7. Suppose that $X_i \subset M$, $i \in I$, are connected sets in a metric space $M$ and $\bigcap_{i \in I} X_i \neq \emptyset$. Prove that then $\bigcup_{i \in I} X_i$ is a connected set.

8. Give an example of a connected union of two disconnected subsets in a metric space.

9. Is the Euclidean space $X \subset \mathbb{R}^2$, given by

$$X = (\{0\} \times [-1, 1]) \cup \{(t, \sin(1/t) \mid 0 < t \leq 1)\} ,$$

connected?

10. Prove that the Euclidean spaces $[0, 1] \times [0, 1] \subset \mathbb{R}^2$ (the unit square in the plane) and $S_1 \subset \mathbb{R}^2$ (the unit circle in the plane) are not homeomorphic.

11. Prove that the Euclidean spaces $\mathbb{R}^2$ and $S_2 \setminus \{(0, 0, 1)\} \subset \mathbb{R}^3$ (the unit sphere in $\mathbb{R}^3$ with the north pole deleted) are homeomorphic.

12. Let $A \subset X \subset M$ be subsets in a metric space $(M, d)$. Prove that the set $A$ is complete in $M$ iff it is complete in the subspace $X$.

13. Prove that any complete subset of a metric space is closed.

14. Prove that any compact subset of a metric space is complete.

15. Prove that any closed subset of a complete metric space is complete.

16. Prove according to the following hints Banach's fixed-point theorem.

    (a) The fixed point is unique: if $x_0, x_1 \in M$ are two fixed points of the map $f$ then $d(x_0, x_1) = 0$.

    (b) If $a_0 \in M, a_1 = f(a_0), a_2 = f(a_1), \ldots$ then for every $n \in \mathbb{N}$ one has $d(a_0, a_n) \leq d(a_0, a_1)(1 + c + c^2 + \cdots + c^{n-1})$. Also, $d(a_{n-1}, a_n) \leq c^{n-1} d(a_0, a_1)$.

    (c) Thus the sequence $(a_n)$ of iterates of the map $f$ starting at $a$ (given in the statement of the theorem) is Cauchy and has the limit $x_0 \in M$.

    (d) But $f$ is continuous (why?), therefore $(f(a_n))$ has the limit $f(x_0)$ and $f(x_0) = x_0$. The theorem is proven.

17. Is the Euclidean subspace $\mathbb{Q} \subset \mathbb{R}$ complete?

18. Is the intersection of two complete sets in a metric space complete?

19. And union?

20. And set difference?