# Lecture 3, October 17, 2019

## Compactness. The Heine–Borel theorem. Relation of a point and a subset. Homeomorphism

**Subspaces.** A *subspace* $X \subset M$ of a metric space $(M, d)$ is the metric space $(X, d)$ with the restricted metric (denoted by the same letter as for the whole space although for $X \neq M$ the two functions are, strictly speaking, different). It is true that a set $A \subset X$ is open (closed) in the subspace $X$ if and only if there is an open (closed) set $Y \subset M$ such that $A = X \cap Y$ (Exercise 10).

One should bear in mind that closedness and openess of a set is a relative notion: for $A \subset X \subset M$ the set $A$ need not be simultaneously open (or closed) both in $M$ and the subspace $X$. Similarly for $A \subset X, Y$ the set $A$ need not be simultaneously open (or closed) in both spaces $X$ a $Y$. For instance, for the Euclidean spaces $A = [0, 1) \subset X = A, Y = [0, 1] \subset \mathbb{R}$ the set $A$ is closed in the space $X$, but not in the space $Y$.

**Continuous maps.** A map $f\colon M \to N$ between two metric spaces is *continuous* if

$$\forall\, a \in M\ \forall\, \varepsilon > 0\ \exists\, \delta > 0 :\ f(B(a,\, \delta)) \subset B(f(a),\, \varepsilon)\ .$$

The first ball is in the space $M$ and the second in $N$. Equivalent definitions of continuity are Heine's (via limits of sequences, see Exercise 1) and the topological one: a map $f\colon M \to N$ between two metric spaces is *topologically continuous* if for every open (closed) set $Y \subset N$, the preimage

$$f^{-1}(Y) = \{a \in M \mid f(a) \in Y\} \subset M$$

is an open (a closed) set in $M$ (Exercise 2).

**Properties of compact sets.** We are still reviewing, as in the case of the topological definition of continuity, material taught in “Matematická analýza II”. Let $(M, d)$ and $(N, e)$ be two metric spaces and let $X \subset M$. If $M$ is compact and $X$ is a closed set, then $X$ is compact (Exercise 3). If $M$ is a general metric space and $X$ is a compact set, then $X$ is a closed and bounded set (Exercise 4). Recall that a set $X \subset M$ is *bounded* if $X \subset B$ for a ball $B$ in $M$. The opposite implication in general does not hold, in the next chapter on sequences and series of functions we give an example of a closed and bounded but non-compact set in a certain metric space of functions, but it does hold in Euclidean spaces (Exercise 5).

If $f\colon M \to N$ is a continuous map and $X \subset M$ is a compact subset, then

$$f(X) = \{f(a) \mid a \in X\} \subset N$$

is compact too (Exercise 6). The maximum-minimum principle holds (Exercise 7): if $f\colon M \to \mathbb{R}$ ($\mathbb{R}$ is the usual Euclidean real line) is a continuous map and $X \subset M$ is a compact subset, then there exist two points $a, b \in X$ such that

$$\forall\, x \in X : \ f(a) \le f(x) \le f(b) \ .$$

Thus $f$ attains on $X$ both its smallest and its largest value. This result is perhaps the main motivation to introduce general abstract definitions of compactness.

**Topological definition of compactness.** We say that a subset $A \subset M$ of a metric space $(M, d)$ is *topologically compact*, if for every system of open sets $X_i$, $i \in I$, in $M$ we have:

$$\bigcup_{i \in I} X_i \supset A \Rightarrow \exists \text{finite } J \subset I : \ \bigcup_{i \in J} X_i \supset A \ .$$

In words and less formally: "every open cover of the set $A$ has a finite subcover". We prove that this definition is equivalent to the original definition of compactness.

**Theorem (Heine–Borel).** *A subset $A \subset M$ of a metric space $(M, d)$ is compact if and only if it is topologically compact.*

*Proof.* Without loss of generality we can take $A = M$ (Exercise 8). We prove the implication $\Rightarrow$ first. Let $(M, d)$ be a compact metric space and

$$M = \bigcup_{i \in I} X_i$$

be its open cover (every set $X_i$ is open); we will find its finite subcover. First we prove that

$$\forall\, \delta > 0\ \exists \text{ a finite set } S_\delta \subset M : \ \bigcup_{a \in S_\delta} B(a,\, \delta) = M \ .$$

For else there would be a $\delta_0 > 0$ and a sequence $(a_n) \subset M$ such that $1 \le m < n \Rightarrow d(a_m, a_n) \ge \delta_0$ — but this contradicts the assumed compactness of $M$ because this sequence has no convergent subsequence (Exercise 9). Indeed, if (we negate the above claim on $\delta$ and $S_\delta$) there existed a $\delta_0 > 0$ such that for every finite set $S \subset M$ one had $M \setminus \bigcup_{a \in S} B(a, \delta_0) \ne \emptyset$, then for already defined points $a_1, a_2, \dots, a_n$ with $d(a_i, a_j) \ge \delta_0$ for every $1 \le i < j \le n$ we could take an $a \in M \setminus \bigcup_{i=1}^n B(a_i, \delta_0)$ and set $a_{n+1} = a$ (this $a = a_{n+1}$ has from each point $a_1, a_2, \dots, a_n$ distance at least $\delta_0$ too). In this way we could define the mentioned sequence $(a_n)$.

For contradiction we now assume that the above open cover of $M$ by the sets $X_i$ has no finite subcover. We claim that it follows from this that (the finite sets $S_\delta$ are defined above)

$$\forall\, n \in \mathbb{N}\, \exists\, b_n \in S_{1/n}\, \forall\, i \in I :\ B(b_n,\, 1/n) \not\subset X_i\ .$$

If it were not so then (we negate the previous claim) there would be an $n_0 \in \mathbb{N}_0$ such that for every $b \in S_{1/n_0}$ there is an $i_b \in I$ with $B(b, 1/n_0) \subset X_{i_b}$. But then, since $M = \bigcup_{b \in S_{1/n_0}} B(b, 1/n_0)$, the indices $J = \{i_b \mid b \in S_{1/n_0}\} \subset I$ give a finite subcover of the set $M$, contrary to our assumption.

The displayed claim on $n$ and $b_n$ therefore holds and we can take the sequence $(b_n) \subset M$. By the assumption it has a convergent subsequence $(b_{k_n}) \subset (b_n)$ with $\lim b_{k_n} = b \in M$. Since the $X_i$ cover $M$, $b \in X_j$ for some $j \in I$. Due to openess of $X_j$ there is an $r > 0$ such that $B(b, r) \subset X_j$. We take $n \in \mathbb{N}$ large enough so that $1/k_n < r/2$ and $d(b, b_{k_n}) < r/2$. For every $x \in B(b_{k_n}, 1/k_n)$ then the triangle inequality implies that $d(x, b) \le d(x, b_{k_n}) + d(b_{k_n}, b) < r/2 + r/2 = r$. Thus

$$B(b_{k_n}, 1/k_n) \subset B(b, r) \subset X_j\ ,$$

in contradiction with the above property of the points $b_n$. The assumption of non-existence of a finite subcover leads to a contradiction, hence the cover of $M$ by the sets $X_i$, $i \in I$, does have a finite subcover.

We prove the implication $\Leftarrow$ which is easier. We assume that every open cover of the set $M$ has a finite subcover, and deduce that every given sequence $(a_n) \subset M$ has a convergent subsequence. First we show that the assumption

$$\forall\, b \in M\ \exists\, r_b > 0 :\ M_b := \{n \in \mathbb{N} \mid a_n \in B(b,\, r_b)\} \text{ is a finite set}$$

leads to a contradiction. Indeed, from the open cover $M = \bigcup_{b \in M} B(b,\, r_b)$ we would choose a finite subcover given by a finite set $N \subset M$ and we would deduce that there is an $n_0$ with $n > n_0 \Rightarrow a_n \notin \bigcup_{b\in N} B(b, r_b)$ because the set of indices $\bigcup_{b\in N} M_b$ is finite (it is a finite union of finite sets). But this is a contradiction because $\bigcup_{b\in N} B(b, r_b) = M$. The assumption therefore does not hold and on the contrary (we again take a negation) it is true that

$$\exists\, b \in M\ \forall\, r > 0:\ M_r := \{n \in \mathbb{N} \mid a_n \in B(b,\, r)\} \text{ is an infinite set .}$$

Now it is easy to select in $(a_n)$ a convergent subsequence $(a_{k_n})$ with the limit $b$. Suppose we have already defined indices $1 \le k_1 < k_2 < \cdots < k_n$ such that $d(b, a_{k_i}) < 1/i$ for $i = 1, 2, \ldots, n$. The index set $M_{1/(n+1)}$ is infinite and therefore there is a $k_{n+1} \in \mathbb{N}$ such that $k_{n+1} > k_n$ and $k_{n+1} \in M_{1/(n+1)}$. Then also $d(b, a_{k_{n+1}}) < 1/(n+1)$. In this way we define a subsequence $(a_{k_n})$ converging to $b$. $\square$

**Relation of a point and a subset.** Let $(M, d)$ be a metric space, $a \in M$, and $X \subset M$. In the following definitions the symbol $U$ denotes a *neighborhood of the point* $a$, an open set $U \subset M$ with $a \in U$. We say that $a$ is

- an *interior point* of the set $X$ if $\exists\, U$ with $U \subset X$.
- an *exterior point* of the set $X$ if $\exists\, U$ with $U \subset M \setminus X$.
- a *boundary point* of the set $X$ if $\forall\, U : U \cap X \ne \emptyset \ne U \cap (M \setminus X)$.
- a *limit point* of the set $X$ if for $\forall\, U$ the intersection $U \cap X$ is infinite.
- an *isolated point* of the set $X$ if $\exists\, U$ such that $U \cap X = \{a\}$.

The interior and isolated points of the set $X$ lie in it and the exterior points lie outside it. The boundary and limit points may lie both in $X$ and outside it. An example illustrating these notions is in Exercise 11.

**Homeomorphism.** Let $(M, d)$ and $(N, e)$ be metric spaces and $f\colon M \to N$ be a map. We say that $f$ is a *homeomorphism* if it is a bijection and both maps $f$ and $f^{-1}$ are continuous. Two spaces are *homeomorphic* if there is a homeomorphism between them. Homeomorphic spaces cannot be distinguished only by open sets. For instance, the Euclidean spaces $(a, b)$ (where $a < b$ are real numbers) and the whole $\mathbb{R}$ are homeomorphic (Exercise 12). For $(a, b) = (-\pi/2, \pi/2)$ the function $\tan x\colon (a, b) \to \mathbb{R}$ is a homeomorphism. On the other hand, the map

$$f\colon [0,\, 2\pi) \to S_1 = \{(x,\, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \subset \mathbb{R}^2,\ f(t) = (\cos t,\, \sin t)\ ,$$

between two Euclidean spaces is not their homeomorphism. It is a continuous bijection, but its inverse $f^{-1}$ is not continuous at the point $(1, 0)$. In the next lecture we prove that the two spaces are in fact non-homeomorphic. The next proposition is closely related to the previous example.

**Proposition (compactness and homeomorphism).** *If $f\colon M \to N$ is a continuous and injective map between metric spaces and $M$ is compact, then the inverse map $f^{-1}\colon f(M) \to M$ is continuous. The map $f$ is therefore a homeomorphism between the spaces $M$ and $f(M)$ (where the last one is given as a subspace of the space $N$).*

We prove it carefully next time.

### Exercises

1. State Heine's definition of continuity and prove that it is equivalent with the original definition.
2. Prove that both versions of topological definition of continuity, with open and with closed sets, are equivalent with the original definition.
3. Prove that every closed subset of a compact space is compact.
4. Prove that every compact subset of a metric space is closed and bounded.
5. Recall the proof of the theorem: if $X \subset \mathbb{R}^n$ is a closed and bounded set in an Euclidean space $\mathbb{R}^n$ then $X$ is compact.
6. Prove that the image of a compact set by a continuous map is a compact set.
7. Recall the proof of the maximum-minimum principle.
8. Why can we in the proof of the Heine–Borel theorem restrict to the case of the whole space?
9. Why a sequence $(a_n) \subset M$ satisfying $d(a_m, a_n) \geq \delta_0 > 0$ whenever $m \neq n$ has no convergent subsequence?
10. Prove the characterization of open and closed sets in a subspace via open and closed sets of the whole space.

11. Let $(M, d) = (\mathbb{R}^2, d_2)$ and $X \subset M$ be given by $X = \{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 < 1\} \cup \{(4, 4)\}$. Describe the interior, exterior, border, limit, and isolated points of the set $X$.

12. Prove that the Euclidean spaces $(a, b)$, $a < b$ are real, and $\mathbb{R}$ are homeomorphic.