Lecture 11, December 12, 2019

Chapter 3: Introduction to complex analysis. Holomorphic and analytic functions. Four differences. Integration

The complex numbers. In the remaining three lectures we prove (in Theorem 1 below): if a function $f\colon \mathbb{C} \to \mathbb{C}$ has derivative everywhere, it is a sum of a power series — there exist complex coefficients $a_0, a_1, \dots$ such that for every $z \in \mathbb{C}$, $f(z) = \sum_{n \geq 0} a_n z^n$.

The complex numbers

$$\mathbb{C} = \{z = a + bi \mid a,\, b \in \mathbb{R}\},\ i = \sqrt{-1}\ ,$$

form a normed field $\mathbb{C} = (\mathbb{C}, 0, 1, +, \cdot, |\dots|)$ (see lecture 2). It has the operations

$$\begin{aligned} (a+bi) + (c+di) &= (a+c) + (b+d)i \\ (a+bi) \cdot (c+di) &= (ac-bd) + (ad+bc)i \end{aligned}$$

and the norm $|z| = |a+bi| = \sqrt{a^2+b^2}$, denoted just by absolute value (Exercise 1). Thus $\mathbb{C}$ is a metric space $(\mathbb{C}, d)$, $d(z_1, z_2) = |z_1 - z_2|$, that is isometric to the classical Euclidean plane $\mathbb{R}^2$ and is *complete*.[1] Symbols $U, U', U_1, \dots$ denote nonempty and open subsets of $\mathbb{C}$ and z denotes the complex variable. We remind the notation

$$\mathrm{re}(a+bi) = a \ \text{ and } \ \mathrm{im}(a+bi) = b$$

for the real and imaginary part of a complex number $a+bi$ and $B(z,r) = \{u \in \mathbb{C} \mid |u-z| < r\}$ for the ball with center z and radius $r > 0$.

Holomorphic and analytic functions. For a function $f\colon U \to \mathbb{C}$ and a point $z_0 \in U$, the *derivative* $f'(z_0)$ *of* f *at* z_0 is defined as for real functions: it is

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \in \mathbb{C}$$

if this limit exists. Explicitly, $f'(z_0) \in \mathbb{C}$ is the derivative of f at z_0 if and only if

$$\forall \varepsilon > 0\, \exists \delta > 0:\ z \in U \ \&\ 0 < |z - z_0| < \delta \Rightarrow \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon\ .$$

[1] Completeness of $\mathbb{C}$ is needed at the latest in the next lecture for the proof of the Cauchy–Goursat theorem.

Definition (holomorphic functions). *A function* $f\colon U \to \mathbb{C}$ *is* holomorphic (on U) *if it has derivative at every point* $z_0 \in U$.

An *entire function* is a function $f\colon \mathbb{C} \to \mathbb{C}$ holomorphic on the whole complex plane $\mathbb{C}$.

Algebraic properties of complex derivatives are the same as in the real case, and we collect them in the next proposition. Its proof is left to the reader as Exercises 2–4.

Proposition (properties of derivatives). *Let* $f, g\colon U \to \mathbb{C}$ *and* $h\colon U' \to \mathbb{C}$ *be holomorphic functions and* $\alpha, \beta \in \mathbb{C}$ *be numbers.*

1. *(linearity) The linear combination* $\alpha f + \beta g$ *is holomorphic and* $(\alpha f + \beta g)' = \alpha f' + \beta g'$.
2. *(the Leibniz rule) The product* fg *is holomorphic and* $(fg)' = f'g + fg'$.
3. *(derivative of a ratio) If* $g \neq 0$ *on* U *then the ratio* $\frac{f}{g}$ *is holomorphic and* $(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$.
4. *(the chain rule) If* $h(U') \subset U$ *then the composite function* $f(h)\colon U' \to \mathbb{C}$ *is holomorphic and* $(f(h))' = f'(h)h'$.

Of course, also $z' = 1$ and $c' = 0$ for every constant $c \in \mathbb{C}$. Thus every rational function, a ratio $\frac{p(z)}{q(z)}$ of two polynomials, is holomorphic on its definition domain and has the same derivative as in the real case.

Definition (analytic functions). *A function* $f\colon U \to \mathbb{C}$ *is* analytic (on U) *if for every point* $z_0 \in U$ *there exist numbers* $a_0, a_1, \dots \in \mathbb{C}$ *such that*

$$z \in U \ \&\ B(z_0, |z - z_0|) \subset U \Rightarrow f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Thus an analytic function is in every ball centered at z_0 and contained in the definition domain expressed by a complex power series with center z_0. We compute with complex power series as with the real ones (see the last and the last but one lecture). It is clear (Exercise 5) that every analytic function is holomorphic. A deep result in complex analysis is the opposite implication: the two above defined classes of functions coincide.

Four differences between real and complex analysis. We explain the most important differences for analysis in the two domains.

Theorem 1 (holomorphic $\Rightarrow$ analytic). *If $f\colon \mathbb{C}\to\mathbb{C}$ is entire then there exist complex coefficients $a_0, a_1, \dots$ such that for any $z\in\mathbb{C}$,*

$$f(z)=\sum_{n=0}^{\infty} a_n z^n\ .$$

We prove it for simplicity only for entire functions, but the more general and already mentioned version holds: if $f\colon U\to\mathbb{C}$ is holomorphic then f is analytic. It follows immediately that every entire (or more generally holomorphic) function has derivatives of all orders.

As we know, and this is the *first difference*, in the real domain Theorem 1 does not hold. The function $f\colon \mathbb{R}\to\mathbb{R}$, given as $f(x)=0$ for $x\le 0$ and $f(x)=x^2$ for $x\ge 0$, has on $\mathbb{R}$ derivative $f'(x)=0$ for $x\le 0$ and $f'(x)=2x$ for $x\ge 0$, but f cannot be expressed on any neighborhood of zero $(-\delta,\delta)$, $\delta>0$, as a sum of power series because $f''(0)$ does not exist.

In the remaining three lectures we also prove the next result.

Theorem 2 (J. Liouville, 1847). *If a function $f\colon \mathbb{C}\to\mathbb{C}$ is entire and bounded then f is constant.*

In the real domain this does not hold at all, and we have the *second difference*. Functions such as $\sin x$, $\cos x$ or e^{-x^2} go from $\mathbb{R}$ to $\mathbb{R}$, have derivatives of all orders and are bounded, but are far from constant (Exercises 6 and 7). The author of the theorem is the French mathematician *Joseph Liouville (1809–1882).*

Corollary (The Fundamental Theorem of Algebra). *If $p(z)=a_nz^n+\dots+a_1z+a_0$ is a polynomial with complex coefficients such that $p(z)\ne 0$ for every $z\in\mathbb{C}$, then $p(z)$ is a constant nonzero polynomial.*

Proof. Let $p(z)$ be as stated. By Exercise 8 there is a $z_0\in\mathbb{C}$ such that for every $z\in\mathbb{C}$, $|p(z)|\ge|p(z_0)|>0$—$|p|$ attains on $\mathbb{C}$ a minimum value. Hence the reciprocal function $1/p(z)$ is everywhere defined and bounded, $|1/p(z)|\le 1/|p(z_0)|$. It is also entire:

$$\left(\frac{1}{p(z)}\right)'=\frac{-p'(z)}{p(z)^2}\ .$$

By the previous theorem, $1/p(z)$ is constant. Thus $p(z)$ is constant. $\square$

The *third difference* between real and complex functions concerns continuity of derivative. The function $f\colon \mathbb{R} \to \mathbb{R}$ given as $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$ has derivative everywhere but f' is discontinuous at $x = 0$ (Exercise 9). No such entire function exists.

Theorem 3 (continuity of complex derivatives). *The derivative $f'\colon \mathbb{C} \to \mathbb{C}$ of a function $f\colon \mathbb{C} \to \mathbb{C}$ is always continuous.*

This is an immediate corollary of Theorem 1 and of course holds more generally for holomorphic functions $f\colon U \to \mathbb{C}$ (Exercise 14).

The *fourth difference* between analysis in $\mathbb{R}$ and analysis in $\mathbb{C}$ concerns local maxima of $|f|$. The function $f(x) = 1-x^2\colon \mathbb{R} \to \mathbb{R}$ has derivatives of all orders, is in fact a polynomial, and has the property that $|f|$ attains at $x = 0$ a strict local maximum: $x \in (-1, 1), x \neq 0 \Rightarrow |f(x)| = 1 - x^2 < 1 = |f(0)|$. No such holomorphic function exists.

Theorem 4 (the maximum modulus principle). *Let $f\colon U \to \mathbb{C}$ be holomorphic. Then*

$$\forall\, a \in U\ \forall\, \delta > 0\ \exists\, b \in U:\ 0 < |b - a| < \delta\ \&\ |f(b)| \geq |f(a)|\ .$$

If f is holomorphic, its modulus $|f|$ has no strict local maximum. We will not prove Theorem 4, but in the remaining three lectures prove all Theorems 1–3.

Segments and rectangles. For it we need an integral over a segment and over the boundary of a rectangle. Let us define these geometric objects. For two different points $a, b \in \mathbb{C}$, the *(straight) segment* $u = ab \subset \mathbb{C}$ is the image

$$u = ab = \{\varphi(t) \mid 0 \leq t \leq 1\}$$

of the interval $[0, 1]$ by the linear function

$$\varphi(t) = (b - a)t + a\colon [0,\, 1] \to \mathbb{C}\ .$$

A segment is oriented by the order of its endpoints, $ab \neq ba$ are two different segments. It has *length* $|u| = |ab| = |b - a| \geq 0$. A *partition* p of $u = ab$ is a $k+1$-tuple $p = (a_0, a_1, \ldots, a_k) \subset u$, $k \in \mathbb{N}$, of points

$$a_i = \varphi(t_i),\ i = 0, 1 \ldots, k\ ,$$

of u that are images of the points $0 = t_0 < t_1 < \cdots < t_k = 1$ in a partition of $[0, 1]$. Thus $a_0 = a$, $a_k = b$ and the points $a_0, a_1, \ldots, a_k$ run on u from a to b. The *norm* $\|p\|$ of p is

$$\|p\| := \max_{1 \le i \le k} |a_{i-1}a_i| = \max_{1 \le i \le k} |a_i - a_{i-1}| \ ,$$

the maximum length of a subsegment in p. Clearly (Exercise 10)

$$\sum_{i=1}^{k} |a_{i-1}a_i| = \sum_{i=1}^{k} |a_i - a_{i-1}| = |a_k - a_0| = |b - a| = |ab| = |u| \ .$$

For a function $f\colon u \to \mathbb{C}$ and a partition $p = (a_0, a_1, \ldots, a_k)$ of the segment u we define the *Cauchy sum* $C(f, p)$ and its *modification* $C'(f, p)$ by

$$\begin{aligned} C(f,\, p) &:= \sum_{i=1}^{k} f(a_i) \cdot (a_i - a_{i-1}) \in \mathbb{C} \ \text{ and} \\ C'(f,\, p) &:= \sum_{i=1}^{k} f(a_{i-1}) \cdot (a_i - a_{i-1}) \in \mathbb{C} \ . \end{aligned}$$

They resemble Riemann sums in Riemann integration. It follows (Exercise 11) that

$$|C(f, p)| = \left| \sum_{i=1}^{k} f(a_i) \cdot (a_i - a_{i-1}) \right| \le \sup_{z \in u} |f(z)| \cdot |u|$$

and the same holds for $C'(f, p)$. We use $C'(f, p)$ to prove that the reversal of a segment changes the sign of the integral.

A *rectangle* $R \subset \mathbb{C}$ is the set

$$R = \{z \in \mathbb{C} \mid \alpha \le \mathrm{re}(z) \le \beta \ \&\ \gamma \le \mathrm{im}(z) \le \delta\}$$

determined by the real numbers $\alpha < \beta$ and $\gamma < \delta$. Its sides are parallel to the real or imaginary axis. If $\beta - \alpha = \delta - \gamma$ then R is a *square*. The *canonic vertices of* R are $(a, b, c, d) \in \mathbb{C}^4$ where

$$a = \alpha + \gamma i, \ \ b = \beta + \gamma i, \ \ c = \beta + \delta i, \ \ \text{and} \ \ d = \alpha + \delta i \ .$$

They begin with the lower left vertex and go anti-clockwisely. The *boundary* ∂R of R is the union of segments

$$\partial R := ab \cup bc \cup cd \cup da \ .$$

The term agrees with the notion of boundary points in metric spaces. The *interior* $\mathrm{int}(R)$ of R is

$$\mathrm{int}(R) := R \setminus \partial R$$

which again agrees with the notion of interior points in metric spaces. The *perimeter* $\mathrm{per}(R)$ *of* R is the sum of lengths of the sides,

$$\mathrm{per}(R) := |ab| + |bc| + |cd| + |da| \ .$$

Integration is the key notion in the proof. Our definition differs somewhat from the standard textbook one because we adapted it for as straightforward proof of Theorems 1–3 as possible.

Definition ($\int$). *Let $f\colon u, \partial R \to \mathbb{C}$ be a continuous function defined on a segment u or on the boundary of a rectangle R. We define*

$$\int_u f := \lim_{n\to\infty} C(f,\, p_n) \in \mathbb{C} \ \ \textit{and} \ \ \int_{\partial R} f := \int_{ab} f + \int_{bc} f + \int_{cd} f + \int_{da} f \in \mathbb{C} \ .$$

Here (p_n) is any sequence of partitions p_n of u with $\lim \|p_n\| = 0$ and (a,b,c,d) are the canonic vertices of R. We call $\int_u f$ the integral of f over the segment u*, and $\int_{\partial R} f$ the* integral of f over the boundary ∂R*.*

Continuity of f ensures that $\int_u f$, and hence $\int_{\partial R} f$, exists. We prove it now.

Theorem (properties of $\int$). *Let $u = ab$, R, and functions f, g be as in the definition. The limit defining $\int_u$ always exists and does not depend on the sequence (p_n). Thus $\int_{\partial R}$ is always defined too. Both integrals have the following properties.*

1. *Linearity: for every $\alpha, \beta \in \mathbb{C}$,*

$$\int_u (\alpha f + \beta g) = \alpha \int_u f + \beta \int_u g$$

and the same holds for $\int_{\partial R}$.

2. *ML bounds — the abbreviation means the maximum modulus of the function times the length of the integration path —*

$$\left|\int_u f\right| \le \max_{z\in u}|f(z)|\cdot|u| \ \ \textit{and} \ \ \left|\int_{\partial R} f\right| \le \max_{z\in\partial R}|f(z)|\cdot \mathrm{per}(R)$$

($\max|f(z)|$ *exist by continuity of* f *and compactness of* u *and* ∂R*).*

3. *Additivity: for every inner point* c *of the segment* $u = ab$*, so* $c \in ab$ *and* $c \ne a, b$*, we have*

$$\int_{ab} f = \int_{ac} f + \int_{cb} f \ . \ \ \textit{Also,} \ \ \int_{ba} f = -\int_{ab} f \ .$$

Proof. Let[2] $f\colon u \to \mathbb{C}$ be a continuous function. It suffices to prove (Exercise 12) that $\forall\,\varepsilon > 0\ \exists\,\delta > 0$ such that for any two partitions p and q of the segment u with $\|p\|, \|q\| < \delta$,

$$|C(f,\,p) - C(f,\,q)| < \varepsilon \ .$$

This Cauchy condition for p and q follows from the uniform continuity of f, which is a corollary of the continuity of f and compactness of u. For the proof of the condition we take for the given $\varepsilon > 0$ a $\delta > 0$ such that

$$x,\,y \in u,\ |x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{|u|} \ .$$

Let $p = (a_0, a_1, \dots, a_k)$ and $q = (b_0, b_1, \dots, b_l)$ be two partitions of u with $\|p\|, \|q\| < \delta$, and we assume in addition that p refines q: $b_j = a_{i_j}$, $j = 0, 1, \dots, l$, for some indices $0 = i_0 < i_1 < \cdots < i_l = k$. Then

$$C(f,\,p) = \sum_{j=1}^{l} C(f,\,p_j) \ ,$$

where $p_j = (a_{i_{j-1}}, a_{i_{j-1}+1}, \dots, a_{i_j})$ is the partition of the segment $u_j = a_{i_{j-1}}a_{i_j} = b_{j-1}b_j$, and

$$C(f,\,q) = \sum_{j=1}^{l} C(g_j,\,p_j)$$

[2] In the lecture I skipped the proof.

where $g_j\colon u_j \to \mathbb{C}$ denotes the function that on u_j has the constant value $f(a_{i_j}) = f(b_j)$ (Exercise 13). By the triangle inequality, the definition of Cauchy sums, the definition of functions g_j, the choice of partitions p and q, and Exercise 10,

$$\begin{aligned}
|C(f,\, q) - C(f,\, p)| &\le \sum_{j=1}^{l} |C(g_j,\, p_j) - C(f,\, p_j)| \\
&= \sum_{j=1}^{l} \left| \sum_{m=a_{i_{j-1}}+1}^{a_{i_j}} (f(a_{i_j}) - f(a_m)) \cdot (a_m - a_{m-1}) \right| \\
&< \sum_{j=1}^{l} \sum_{m=a_{i_{j-1}+1}}^{a_{i_j}} \frac{\varepsilon}{|u|} |a_m - a_{m-1}| = \sum_{j=1}^{l} \frac{\varepsilon}{|u|} |b_j - b_{j-1}| \\
&= \frac{\varepsilon}{|u|} |u| = \varepsilon\ .
\end{aligned}$$

In the general case we use the trick of a common refinement. For the given $\varepsilon > 0$ we take the $\delta > 0$, whose existence we proved in the previous paragraph, such that for any two partitions p' and q' of u such that $\|p'\|, \|q'\| < \delta$ and one of them refines the other one has $|C(f, p') - C(f, q')| < \frac{\varepsilon}{2}$. If p and q are any partitions of u with $\|p\|, \|q\| < \delta$, we take their common refinement, the partition $r = p \cup q$. It refines both p and q and $\|r\| < \delta$. By the definition of δ we have the desired inequality

$$|C(f,p) - C(f,q)| \le |C(f,p) - C(f,r)| + |C(f,r) - C(f,q)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\ .$$

This proves the existence of $\int_u f$ and $\int_{\partial R} f$.

To prove the properties 1–3 by the limit transition $n \to \infty$ is easy. The linearity follows from the linearity of Cauchy sums $C(\cdot, p_n)$ in the first variable. The first ML bound follows from the estimate of Cauchy sums in Exercise 11, and the second folows from the first. The first identity in additivity follows from the additivity of Cauchy sums in the second variable: $C(f,p) = C(f,q) + C(f,r)$ if q and r are partitions of the segments ac and cb, respectively, and $p = qr$ is the concatenated partition of ab; $\|p\| = \max(\|q\|, \|r\|)$. The second identity follows from the fact that $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ such that for every partition p of $u = ab$ with $\|p\| < \delta$ one has $|C(f,p) - C'(f,p)| < \varepsilon$ (a corollary of the uniform continuity of f), and the fact that for every partition $p = (a_0, a_1, \ldots, a_k)$ of ba one has

$$C(f,\, p) = \sum_{i=1}^{k} f(a_i)(a_i - a_{i-1}) = -\sum_{i=1}^{k} f(a_i)(a_{i-1} - a_i) = -C'(f,\, p')$$

where $p' = (a'_0, a'_1, \ldots, a'_k) = (a_k, a_{k-1}, \ldots, a_0)$ is the partition of ab arising from p by the reversal of ba. Of course, $\|p\| = \|p'\|$. □

Exercises

1. Prove that $\mathbb{C} = (\mathbb{C}, 0, 1, +, \cdot, |\ldots|)$ is a complete normed field.
2. Prove the linearity of differentiation and the Leibniz rule.
3. Prove the formula for the derivative of a ratio.
4. Prove the chain rule.
5. Prove that analytic functions have derivatives of all orders.
6. Give examples of infinitely many real functions that are everywhere defined, are arbitrarily differentiable and bounded, but are not constant.
7. Prove that for every k-tuple of pairs $(a_i, b_i) \in \mathbb{R}^2$, $i = 1, 2, \ldots, k$, there is a real, everywhere defined, arbitrarily differentiable and bounded function f such that $f(a_i) = b_i$ for every $i = 1, 2, \ldots, k$.
8. Prove that for every complex polynomial $p(z)$ the function $|p(z)|$ attains on $\mathbb{C}$ a minimum value.
9. Compute the derivative of the function $f\colon \mathbb{R} \to \mathbb{R}$ given as $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that f' is discontinuous at 0.
10. Prove that the sum of lengths of the subsegments in a partition of a segment u is exactly the length $|u|$.
11. Prove that the absolute value of a Cauchy sum is at most the supremum of absolute values of values of the function on the segment times the length of the segment.
12. Show that if the stated Cauchy condition for p and q holds then the limit defining $\int_u$ exists and is independt on the sequence (p_n).

13. Explain the identity $C(f,\, q) = \sum_{j=1}^{l} C(g_j,\, p_j)$ in the last proof.
14. Deduce Theorem 3 from Theorem 1.