Matematické struktury tutorial 9 on April 24, 2017: separation axioms $T_0, T_1, T_2, T_3, T_{3\frac{1}{2}}$, and T_4

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I went through most of section 5, pp. 105–111, in Chapter 5 "Topologie" of the lecture notes

- A. Pultr, *Matematické struktury*, 2005, 155 pp., available at `http://kam.mff.cuni.cz/~pultr/`

What remains to do: to prove Theorem 5.5.2, to prove Urysohn's lemma, discuss separation axioms versus subspace and product and to state and prove Tichonov's embedding theorem.

What is not in the lecture notes but what I mentioned is the *Zariski topology* which is a fundamental notion in algebraic geometry. For $P \subseteq \mathbb{C}[x_1, x_2, \ldots, x_n]$ (a set of polynomials with complex coefficients) we denote their common zeros by

$$Z(P) := \{\overline{a} \in \mathbb{C}^n \mid f(\overline{a}) = 0 \ \forall f \in P\} .$$

Note that $Z(P) = Z(\langle P \rangle)$ where $\langle P \rangle$ (never write $< P >$!) is the ideal in the ring $\mathbb{C}[x_1, x_2, \ldots, x_n]$ generated by the set P, that is, $\langle P \rangle = \{p_1 f_1 + \cdots + p_k f_k \mid f_i \in P, p_i \in \mathbb{C}[x_1, x_2, \ldots, x_n]\}$.

Exercise. *Show that the set system*

$$C = \{Z(P) \mid P \subseteq \mathbb{C}[x_1, x_2, \ldots, x_n]\}$$

satisfies the axioms of closed sets of a topology (contains $\emptyset$ and $\mathbb{C}^n$ and is closed to any intersections and to finite unions).

The topology defined by the closed sets C is exactly the Zariski topology on $\mathbb{C}^n$. It is not T_2 — one shows without much trouble that any two nonempty open sets of this topology intersect. In fact, we get the same set system C by taking just finite sets P — as D. Hilbert proved, every ideal in the ring $\mathbb{C}[x_1, x_2, \ldots, x_n]$ is finitely generated.

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