Matematické struktury
tutorial 5 on March 20, 2017: a graph-theoretic proof of the Cantor–Bernstein theorem, Kalmár's theorem on games and a characterization of modular lattices

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For two injections $f : X \to Y$ and $g : Y \to X$ we show by means of two graphs that there is a bijection $h : X \to Y$ that, moreover, satisfies that $h(x) = f(x)$ or $h(x) = g^{-1}(x)$. We may assume that $X \cap Y = \emptyset$.

Exercise. *Show how to reduce the general case when X and Y may intersect (indeed, we may have $X = Y$) to the case when they are disjoint.*

For $V = X \cup Y$ we consider the oriented graph $D = (V, f \cup g)$ (here $f, g \subseteq V \times V$) and the unoriented graph $G = (V, E)$ where $E \subseteq \binom{V}{2}$ arises from $f \cup g$ simply by forgetting order in each pair. For $(a, b) \in V \times V$ we denote the fact that $(a, b) \in f \cup g$ by the arrow $a \to b$. Then, clearly, for every $a \in V$ either exactly one arrow leaves a and none enters it (type 1 vertex) and this arrow is either from f or from g, or exactly one arrow both enters and leaves a (type 2 vertex) and one of these arrows is from f and the other from g. This follows from the assumption what f and g are.

We consider a (connected) component $K \subseteq V$ of the graph G. If $a \in K$ for a type 1 vertex a then it follows that

$$K = \{a = v_0 \to v_1 \to v_2 \to \dots\}, \quad \text{all } v_i \text{ are distinct}$$

— K is a one-way infinite oriented path. (The path starting at a cannot return to already visited vertices and there is no vertex in K besides it because of the arrow property of D (and the assumption on a).) If K consists of only type 2 vertices, then a similar argument (to the one in brackets) shows that K is either an oriented finite even cycle

$$K = \{v_0 \to v_1 \to v_2 \to \dots \to v_{2n-1} = v_0\}, \quad n \in \mathbb{N}, \text{ all } v_0, \dots, v_{2n-2} \text{ distinct}$$

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or a two-way infinite oriented path

$$K = \{\cdots \to v_{-2} \to v_{-1} \to v_0 \to v_1 \to v_2 \to \dots\}, \quad \text{all } v_i \text{ are distinct .}$$

We set $M = (V, F)$, $F \subseteq E$, to be the subgraph of G with edges

$$F = \bigcup_K \{\{v_i, v_{i+1}\} \mid i \text{ is an even number}\} .$$

Here K runs through all components of G and the vertices of K are numbered as given above for the three types of K. It follows that M is a perfect X-Y matching: the edges $e \in F$ are disjoint, cover the whole V, and each $e \in F$ contains one vertex from X and one from Y (the f-arrows and g-arrows alternate in every K). We order the pairs in F by putting the elements of X first and get the desired bijection h from X to Y. $\qquad\square$

We formalize the notion of a game with complete information, for the *first player f* and the *second player s* who alternate in moves and f plays first. We are given X, x_0, A, B where X is a set of all possible *states of the game*, $x_0 \in X$ is the *initial state*, and $A, B \subseteq X \times X$ are the *rules of the game* for f and s, respectively. A *(particular) game* is a sequence, finite or infinite, $x_0, x_1, x_2, \dots \in X$ such that

$$x_0 A x_1 B x_2 A x_3 B x_4 \dots \quad (\text{i.e. } x_0 A x_1, x_1 B x_2, \dots) .$$

Thus f opened this game with the move (x_0, x_1), then s made the move (x_1, x_2), f answered with (x_2, x_3) and so on. A player p *loses* and p's opponent *wins* if p has no move available according to the rules. For example, s loses if f has played aAb and $bB = \emptyset$. An infinite game is considered a *draw*. A *strategy* of f (resp. s) is a subset $S \subseteq A$ (resp. $S \subseteq B$). This strategy of f (resp. s) is *persistent* if for any $a, b, c \in X$,

$$aSbBc \Rightarrow \exists\, d \in X: \ cSd \ (\text{resp. } aSbAc \Rightarrow \exists\, d \in X: \ cSd) .$$

Thus once f has played according to a persistent strategy S, f has an answer to any move of s and cannot lose, and similarly for s. A *non-losing strategy* for f (resp. s) is a persistent strategy S such that for any $a \in X$,

$$x_0 S \neq \emptyset \ (\text{resp. } x_0 A a \Rightarrow \exists\, b \in X: \ aSb)$$

— the player can always enter S (and then by playing according to S cannot lose). Finally, a *winning strategy* of a player p is a strategy S that ensures victory for p no matter how the opponent plays.

Theorem 1 (Kalmár, 1928) *At least one player, f or s or both, has a non-losing strategy. Hence if infinite games are not allowed (according to the rules A and B) then exactly one of f and s has a winning strategy.*

Proof. We again apply this particular case of the Tarski–Knaster theorem: every inclusion-wise monotonous mapping from the power set of a set to itself has a fixed point. Let X, x_0, A, B be given. For $P \subseteq X \times X$ we set

$$r(P) = \{(x, y) \in X \times X \mid yP = \emptyset\} \,,$$

these are the moves that cannot be answered by any move from P, and

$$\phi_{AB}(P) = A \cap r(B \cap r(P)), \ \phi_{AB} : \ \exp(X \times X) \to \exp(X \times X) \,.$$

This mapping is monotonous because $r : \ \exp(X \times X) \to \exp(X \times X)$ is anti-monotonous (reverses inclusions) and is applied twice in the definition of ϕ_{AB}. Let $S_{II} \subseteq X \times X$ be a fixed point of the mapping ϕ_{BA} (guaranteed by the T.–K. theorem):

$$B \cap r(A \cap r(S_{II})) = \phi_{BA}(S_{II}) = S_{II}, \ \text{ and let } \ S_I := A \cap r(S_{II}) \,.$$

Then S_I is a fixed point of ϕ_{AB}:

$$\phi_{AB}(S_I) = A \cap r(B \cap r(A \cap r(S_{II}))) = A \cap r(S_{II}) = S_I \,.$$

We claim that S_I is a persistent strategy for f, and S_{II} for s. We show it for s. Let $xS_{II}yAz$. If $zS_{II} = \emptyset$ then $(y, z) \in A \cap r(S_{II})$. But $(x, y) \in S_{II} \subseteq r(A \cap r(S_{II}))$ and $y(A \cap r(S_{II})) = \emptyset$, in contradiction with the fact that z lies in this set.

Exercise. *Prove similarly that S_I is a persistent strategy for f.*

We conclude the proof by showing that S_I is a non-losing strategy for f or S_{II} is a non-losing strategy for s. Suppose that S_{II} is not a non-losing strategy for s: there is an $x_1 \in X$ such that $x_0 A x_1$ and $x_1 S_{II} = \emptyset$. Then $(x_0, x_1) \in A \cap r(S_{II}) = S_I$ and S_I is a non-losing strategy for f. □

Characterization of modular lattices

Recall that a *lattice* is a poset $L = (L, \leq)$ such that any pair of elements $a, b \in L$ has a supremum $a \vee b$ and an infimum $a \wedge b$. We call it *modular* if

$$a, b, c \in L, \ a \leq c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c \,.$$

In every lattice we have $a \vee (b \wedge c) \leq (a \vee b) \wedge c$ because, in this situation, $\{a, b \wedge c\} \leq \{a \vee b, c\}$.

$K \subseteq L$ is a *sublattice* if $a, b \in K \Rightarrow \{a \vee b, a \wedge b\} \subseteq K$ — we can restrict the operations of join and meet to K. Two lattices are *isomorphic* if they are isomorphic as posets (there is a bijection between their groundsets that maps one relation to the other). We define the five-element poset $C_5 = (\{a, b, c, x, y\}, \leq)$ by

$$b < a < c > y > x > b \ \text{ and } a \text{ is incomparable to both } x \text{ and } y \,.$$

C_5 is a lattice but not a modular one: although $x < y$,

$$x \vee (a \wedge y) = x \vee b = x, \ (x \vee a) \wedge y = c \wedge y = y \,.$$

Theorem 2 *A lattice $(L, \leq)$ is modular if and only if it has no sublattice isomorphic to C_5.*

Proof. We have already seen that if L has a sublattice isomorphic to C_5 then L is not modular. We suppose that L is not modular and find in it a copy of C_5. Thus there are $u, v, w \in L$ such that

$$u \leq w \quad \text{and} \quad u \vee (v \wedge w) < (u \vee v) \wedge w \, .$$

We set

$$a = v, \ b = v \wedge w, \ c = v \vee w, \ x = u \vee (v \wedge w) \quad \text{and} \quad y = (u \vee v) \wedge w \, .$$

In the next lecture we show that these five elements are distinct and induce a copy of C_5.