(L9) (March 23, 2020) I just remark that a graph ①
$G=(V,E)$, $E\subseteq\binom{V}{2}$, can be defined also as a binary relation $E\subseteq V\times V$ that is symmetric and ireflexive (never vEv, $v\in V$).

Thus we will prove the (J. Combin. Theory, B 22 (1977), 281-3)

Theorem (V. Müller, 1977)
If $G=(V,E)$ and $H=(W,F)$ are graphs with $n=|V|=|W|\geq 6$ vertices and s.t. $S(G)\cong S(H)$ ($\exists$ a bijection $\beta: E\to F$ s.t. $\forall e\in E: G\setminus e\cong H\setminus\beta(e)$) then $G\cong H$ (G and H are isomorphic).

Proof. We may assume that $V(G)=V(H)=V$. So $|V|=n\geq 6$ and $E,F\subseteq\binom{V}{2}$ with $|E|=|F|=m\geq 1+n(\log_2 n-1)$ (where $\log_2(\cdot)$ denotes the binary logarithm, logarithm to base 2: for real $x>0$, y one has $\log x=y\iff x=2^y$).

For $A\subseteq E$ and $e\in E$, $f\in F$ we consider the number of A-copies in G, and in $G\setminus e$:
$N(G):=\#\{B\subseteq E\mid A\cong B\}$ and

$N_e(G) := \#\{B \subseteq E\setminus\{e\} \mid A \cong B\}$.

Here $A \cong B$ means that $(V, A) \cong (V, B)$ (the two graphs are isomorphic). The numbers $N(H)$ and $N_f(H)$ are defined in an analogous way (F replaces E but A is still a subset of E).

We claim that $\boxed{\forall A \subseteq E,\ A \neq E:\ N(G) = N(H)}$.

(G and H contain ~~the~~ equal numbers of A-copies) proof:

$$(m-|A|)N(G) = \sum_{e\in E} N_e(G) \overset{(*)}{=} \sum_{f\in F} N_f(H) = (m-|A|)N(H).$$

Equality follows from counting the pairs

$$\{(e, K) \mid e \in E,\ K \subseteq E\setminus\{e\},\ K \text{ is an } A\text{-copy}\}$$

– first group the pairs according to K's and then according to ~~to~~ e's (this is one of many proofs in combinatorics by double counting). Same for. The crucial middle equality follows by employing the bijection $\beta: E \to F$ that witnesses $S(G) \cong S(H)$. Namely, $G - e \cong H - \beta(e) = H - f$ for $\forall e \in E$.

Two isomorphic graphs contain the same numbers of copies of any fixed graph L [why? –

Problem 9.1 Let G, H and L be graphs where $G \cong H$ (are isomorphic). Prove that the ~~n~~ number of L-copies in G = the # of L-copies in H.

($G = (V, E)$, then any L-copy in G is any subgraph $G' = (V', E')$, $V' \subseteq V$ and $E' \subseteq E$, s.t. $G' \cong L$).]

and therefore $N_e(G) = N_{f(e)}(H) = N_f(H)$ for $\forall e \in E$. Thus the two sums over E and over F differ only by a permutation of their summands and are equal. Dividing in (*) by $n - |A|$ we get $\boxed{\;\dots\;}$.

Recall the PIE (principle of inclusion and exclusion): If $X_1, \dots, X_n \subseteq X$ are finite sets then ($\bigcap_{\emptyset} X_i = X$)

$$\left| X \setminus \bigcup_{i=1}^{n} X_i \right| = \sum_{I \subseteq [n]} (-1)^{|I|} \left| \bigcap_{i \in I} X_i \right|$$

Problem 9.2

Prove the PIE, double counting arguments are preferred.

We introduce this notation: if K, L are graphs and $D \subseteq E(K)$ then

$\mathrm{inj}(K,L) := \#\{f : V(K) \to V(L) \mid f$ is injective and $\forall e \in E(K)$ we have $f(e) \in E(L)\}$

— the # of injective hom.s from K to L

$\mathrm{inj}_D(K,L) := \#\{f : V(K) \to V(L) \mid f$ is injective and $\forall e \in E(K)$ we have $f(e) \in L \Leftrightarrow e \notin D\}$.

So $\mathrm{inj}(K,L) = \mathrm{inj}_{\emptyset}(K,L)$ and $\mathrm{inj}_D(K,L)$ counts injections from $V(K)$ to $V(L)$ that sends edges in D outside $E(L)$ and edges is $E(K) \setminus D$ to $E(L)$.

For $H = (V, F)$ we denote the complementary graph by $\overline{H}$, so $\overline{H} = (V, \binom{V}{2} \setminus F)$. Recall that $G = (V, E)$, $H = (V, F)$, $m = |E| = |F|$ (where $E, F \subseteq \binom{V}{2}$) and that $S(G) \cong S(H)$. Then PIE gives for $\forall D \subseteq E$ that

$$\mathrm{inj}_D(G, \overline{H}) = \sum_{A \subseteq E \setminus D} (-1)^{|A|} \mathrm{inj}((V, D \cup A), H).$$

$\boxed{\text{Problem 9.3}}$ Prove this equality.

Continuation next time...