(L7) (March 16, 2020) If Δ is a type, we denote by (1)
Rel(Δ) the class of all relational structures of type Δ and all homomorphisms between them.

Substructures Let $(X, R=(R_t)_{t\in T})$ be in Rel(Δ) and $Y \subseteq X$, with $j: Y \to X$ $(j(y)=y)$ denoting the inclusion map.
We define $R|Y = (R_{Y,t})_{t\in T}$, $R_{Y,t} = \{ \xi: \Delta_t \to Y \mid j\xi \in R_t \}$.
We say that the rel. structure

$$(Y, R|Y) \in \mathrm{Rel}(\Delta)$$

is a substructure of (X,R) (induced by the subset Y). In the finitary case:

$$R_{Y,t} = \{ (x_1, x_2, \dots, x_n) \mid (x_1, \dots, x_n) \in R_t \cap (\underbrace{Y \times Y \times \dots \times Y}_{n}) \}.$$

Products Let $(X_i, R_i) \in \mathrm{Rel}(\Delta) = \mathrm{Rel}((\Delta_t)_{t\in T})$, $i \in I$. We define rel~~structure~~ations R on the Cart. product $X = \prod_{i\in I} X_i$ (with the projections $p_i: X \to X_i$) by

$$(\xi: \Delta_t \to X) \in R_t \iff \forall i \in I: p_i \xi \in R_{i,t}.$$

We call the rel. structure (X, R) product of the system (X_i, R_i), $i \in I$, and denote it by $\prod_{i\in I} (X_i, R_i)$.

For example, for $I=[2]$ and $(\Delta_t)_{t\in T} = ([3])$, $|T|=1$, we have in the product $(X_1,R_1)\times(X_2,R_2)$: $R \ni ((x_1,y_1),(x_2,y_2),(x_3,y_3))$ iff $(x_1,x_2,x_3)\in R_1$ & $(y_1,y_2,y_3)\in R_2$. If all $(X_i, R_i) = (X, R)$, $i \in I$, are the same, we write $(X,R)^I$ for $\prod_{i\in I}(X_i,R_i)$. (See p. 6 – again)

Factor-structures (quotients): Let $(X, R) \in Rel(\Delta)$, ~~$i \in I$~~ and $q : X \to Y$ be onto. We define on Y relations $\bar{R}$ of type Δ by $\bar{R}_t = \{ q\xi \mid \xi \in R_t \}$, $t \in T$. We say that the rel. structure $(Y, \bar{R})$ is a *quotient* or a *factor-structure* of (X, R) (according to q, or according to the equivalence $E = \{(x,y) \mid q(x) = q(y)\}$).

Projective and injective constructions of structures: Let $(X_i, R_i) \in \underset{Rel(\Delta)}{Rel((\Delta_t)_{t \in T})}$, $i \in I$, and let some maps $r_i : X \to X_i$, $i \in I$, be such that

$$(\forall i \in I : r_i f = r_i g) \Rightarrow f = g \quad (\text{for any } f, g : Y \to X).$$

Then the rel. structure $\underset{\in Rel(\Delta)}{(X, R')}$, $R'_t = \{ \xi : \Delta_t \to X \mid \forall i \in I : r_i \xi \in R_{i,t} \}$ is the largest rel. str. of type Δ (on X) s.t. all maps r_i are hom-s $(X, R) \to (X_i, R_i)$, and we say that (X, R') is *projectively constructed from the (X_i, R_i), $i \in I$.* Examples are substructures (with just one map $j : Y \subseteq X$) and products (with projections $\prod_{i \in I} X_i \to X_i$).

Similarly, let some maps $s_i : X_i \to X$, $i \in I$, satisfy $(\forall i \in I : f s_i = g s_i) \Rightarrow f = g$ (for any $f, g : X \to Y$). Then the rel. structure $(X, R') \in Rel(\Delta)$, $R'_t = \{ s_i \xi \mid \xi \in R_{i,t}, i \in I \}$ is the smallest rel. structure of type Δ on X s.t. all maps s_i are hom.-s $(X_i, R_i) \to (X, R)$.

We say that (X, R') is <u>injectively constructed</u> from the (X_i, R_i), $i \in I$. So far we have met only one injective construction, of a quotient. Another example:

<u>Sums.</u> Let $(X_i, R_i) \in \mathrm{Rel}(\Delta) = \mathrm{Rel}((\Delta_t)_{t\in T})$, $i \in I$ — for simplicity we assume that the sets X_i are disjoint —, $X = \bigcup_{i\in I} X_i$, and $j_i : X_i \subseteq X$ be the inclusion maps. Then the inj. construction gives on X a rel. structure $R = (R_t = \bigcup_{i\in I} \{ j_i \xi \mid \xi \in R_{i,t}\})_{t\in T}$.
We say that (X, R) is a <u>sum of</u> the rel. structures (X_i, R_i), $i \in I$.
Then all maps $j_i : (X_i, R_i) \to (X, R)$ are hom.-s.

<u>An application of graph homomorphisms: The graph (edge-) reconstruction problem.</u> A graph G is a pair $G = (V, E) = (V(G), E(G))$, $E \subseteq \binom{V}{2} = \{ e \subseteq V \mid |e| = 2\}$, where V is the <u>set of vertices</u> and E is the <u>set of edges</u>.[1)] Two graphs G and H are <u>isomorphic</u> $\Longleftrightarrow$ $\exists$ a bijection $f : V(G) \to V(H)$ s.t. $\forall \{u, v\} \in \binom{V(G)}{2}$: $\{u, v\} \in E(G) \Longleftrightarrow \{f(u), f(v)\} \in E(H)$. *)

For $G = (V, E)$ and $e \in E$ we set $G - e := (V, E \setminus \{e\})$ (<u>deletion</u> of the edge e). The <u>deck of G</u> is ~~[illegible]~~

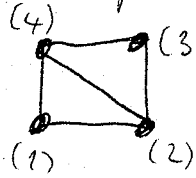
$$\underset{\substack{\| \\ G}}{(V, E)} \quad \underset{\substack{\| \\ H}}{(W, F)} \qquad \boxed{S(G) := \{ G - e \mid e \in E \}}.$$

Two graphs G and H have <u>equal decks</u>, written $S(G) \cong S(H)$, if there is a bijection $\beta : E \to F$ s.t. $\forall e \in E$:

$$G \setminus e \cong H \setminus \beta(e).$$

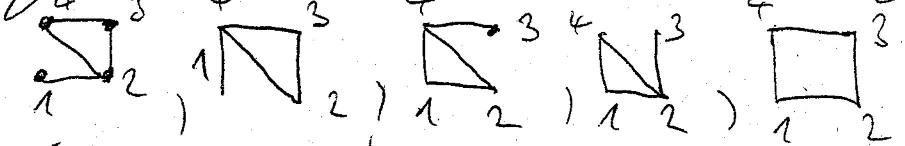
*) Notation: $G \cong H$

1) We consider here only <u>finite</u> graphs

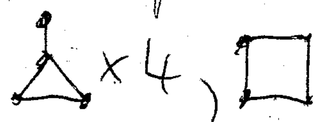
To determine if $S(G) \cong S(H)$, only the isomorphism types of G's are relevant. For example, $G =$  has deck: and H's

~~$S(G) = \{$ ... $\}$ ... so $S(G) \cong S(H)$ if~~

~~$S(G) \neq S(H)$ as as multisets ("sets with possibly repeated elements").~~

$S(G) = \{$  $\}$

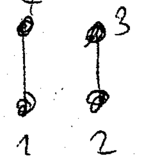
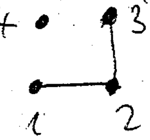
and if we write only isomorphism types, then

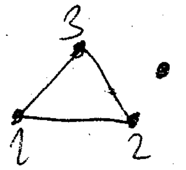
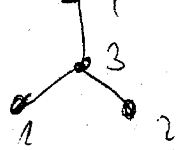
$S(G)_{iso} = \{$  $\}_m$ ← multiset (a set "with possibly repeated elements). If $G = (V, \emptyset)$ has no edge, then trivially $S(G) = \emptyset$.

$\boxed{\text{Problem 7.1.}}$ Prove that if $G = (V, E)$ and $H = (W, F)$ are such that $|E| = |F| = 2$ and $S(G) \cong S(H)$ then $G \cong H$.

But it may happen that $E, F \neq \emptyset$, $S(G) \cong S(H)$ but $G \not\cong H$:

$\boxed{\text{Problem 7.2.}}$ Show that this happens for these two examples:

a) $G =$ , $H =$ 

b) $G =$ , $H =$ 

But larger examples like should not exist; we have the edge-reconstruction conjecture

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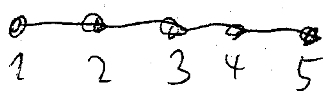
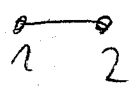
$\boxed{\text{Conjecture (F. Harary, 1964)}}$ If $G = (V, E)$, $H = (W, F)$ are graphs such that $|E|, |F| \geq 4$ and $S(G) \cong S(H)$, then $G \cong H$.

This problem is still unsolved.

We will eventually prove the following result towards Harary's edge-reconstruction conjecture.

Theorem (Müller, 1977) If $G=(V,E)$ and $H=(W,F)$ are graphs with $n=|V|=|W|\geq 6$ vertices and $|E|=|F|=m \geq 1+n(\log_2(n)-1)$ edges and $S(G)\cong S(H)$ then $G\cong H$ (G and H are isomorphic). Thus for graphs with not too few edges the edge-reconstruction conjecture holds. The proof is based on counting homomorphisms. If $G=(V,E)$ and $H=(W,F)$ are graphs, a homomorphism (further on just "hom." or "hom.s") from G to H is any map $f: V\to W$ s.t. $\{u,v\}\in E \Rightarrow \{f(u),f(v)\}\in F$. It is indeed a particular case of a hom. of relational structures. For example if $G=$  and $H=$ , then $f(1)=f(3)=f(5)=1$, $f(2)=f(4)=2$ is a hom. from G to H.

Recall that for a graph $G=(V,E)$, the chromatic number $\chi(G)$ of G is the minimum $k\in\mathbb{N}$ such that there exists a map $c: V\to[k]$ $(=\{1,2,\dots,k\})$ s.t. $\{u,v\}\in E \Rightarrow c(u)\neq c(v)$. $K_k=\left([k],\binom{[k]}{2}\right)$ is the complete graph on k vertices.

Problem 7.3. Prove that $\chi(G)\leq k \iff \exists$ a hom. from G to K_k.

Again and more legibly:

We call the rel. structure (X,R) a product of the system $(X_i,R_i), i\in I$, and denote it by $\prod_{i\in I}(X_i,R_i)$.

For example, for $I=[2]=\{1,2\}$ and $\Delta=(\Delta_t)_{t\in T}=([3])$ (and $T=\{0\}$) we have in the product $(X_1,R_1)\times(X_2,R_2)$:
$((x_1,y_1),(x_2,y_2),(x_3,y_3))\in R_0 \iff (x_1,x_2,x_3)\in R_{1,0}$ and $(y_1,y_2,y_3)\in R_{2,0}$.

If all $(X_i,R_i)=(X,R)$ are the same (equal), we write their product as a power

$$\boxed{\prod_{i\in I}(X,R)=(X,R)^I.}$$