Math. Structures (L 5) (March 9, 2020)

A set X is <u>countable</u> if $X \simeq \mathbb{N} = \{1, 2, \dots\}$ and is <u>finite</u> if $X \simeq [n] = \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}_0 = \{0, 1, \dots\}$. X is <u>at most countable</u> if it is finite or countable. E.g., the set of fractions $\mathbb{Q}$ is countable.*)

Problem 5.1. Prove: $X_1, X_2, \dots$ are countable $\Rightarrow$ $\bigcup_{n=1}^{\infty} X_n$ is countable.

Problem 5.2 Prove: If $X \subseteq \mathbb{R}$ is s.t. $(X, \leq)$ (the standard ordering of $\mathbb{R}$) is a well ordering, then X is at most countable. Hint: find an injection $f: X \to \mathbb{Q}$.

*) $\mathbb{R}$ is uncountable

The prophet paradox is another paradoxical corollary of the AC. For $f: \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$ we denote by $f \mid (-\infty, a)$ the restriction of f on the interval.

Theorem (the PP) Under the assumption of the AC there exists a mapping

$$V: \{g \mid g: (-\infty, a) \to \mathbb{R},\ a \in \mathbb{R}\} \to \mathbb{R}$$

s.t. for every function $f: \mathbb{R} \to \mathbb{R}$ the set

$$\{a \in \mathbb{R} \mid V(f \mid (-\infty, a)) \neq f(a)\}$$

is at most countable. ∎

②

In other words, a prophet V exists that is able for any function $f:\mathbb{R}\to\mathbb{R}$ to guess correctly its value $f(a)$ from the values $f(b)$, $b<a$, and errs for only at most countably many a's.

Proof. $\mathfrak{M}:=\{f \mid f:\mathbb{R}\to\mathbb{R}\}$. By the Zermelo thm. there is a well ordering $\langle\mathfrak{M},\preceq\rangle$. For $g:(-\infty,a)\to\mathbb{R}$ we set $V(g):=f_0(a)$ where $f_0=\min_{\preceq}(\{f\in\mathfrak{M} \mid f|(-\infty,a)=g\})$. Now let a function $f\in\mathfrak{M}$ be given and $X=\{a\in\mathbb{R} \mid V(f|(-\infty,a))\neq f(a)\}$ be the set of errors of V.

Book: Hardin–Taylor, The math. of coordinated inference, Springer, 2013

We prove the implication: $a,b\in\mathbb{R}$, $a<b$, $a\in X$, $g_a:=\min_{\preceq}(\underbrace{\{g\in\mathfrak{M} \mid g|(-\infty,a)=f|(-\infty,a)\}}_{\mathfrak{M}_a})$, $g_b:=$ similarly, then $g_a\prec g_b$.

Indeed, $a<b\Rightarrow \mathfrak{M}_b\subseteq\mathfrak{M}_a\Rightarrow g_a\preceq g_b$. But $g_a(a)=V(f|(-\infty,a))\neq f(a)=g_b(a)$ and $g_a\neq g_b$.

Thus $(X,\leq)$ is well ordered (else an infinite strictly descending chain in $(X,\leq)$ would give the same chain in $(\mathfrak{M},\preceq)$ which is not possible as $(\mathfrak{M},\preceq)$ is a well ordering). By Pr. 5.2, X is at most countable. $\boxtimes$

A set or class X is <u>transitive</u> if $a \in b \in X \Rightarrow a \in X$. (3)

For example, the natural numbers of set theory $0 = \emptyset$, $1 = \{0\}$, $2 = \{0, \{0\}\}$, ... are tr. sets.

An <u>ordinal</u> (or <u>ordinal number</u>) is a set X that is well ordered by $\in$ and is transitive. All ordinals:

$\mathrm{Ord} := \{\boxed{0}, \boxed{1}, \boxed{2}, \dots; \boxed{\omega}, \omega+1, \omega+2, \dots; 2\omega, 3\omega, \dots; \omega^2, \dots; \omega^\omega, \dots; \omega^{\omega^\omega}, \dots; \boxed{\omega_1}, \dots\}$ <u>is a proper class</u>

ω: 1st infinite ordinal

ω_1: 1st uncountable ordinal

- Ord is trans. class and is <u>not</u> a set (else R. par.)
- It is the only proper class well-ordered by $\in$
- Every well-ordered set $(X, \leq)$ is isomorphic to exactly one ordinal $\alpha = (\alpha, \in)$ — the order type of X.

$\alpha \in \mathrm{Ord}$ is a <u>cardinal</u> (or <u>cardinal number</u>) if: $\beta \in \mathrm{Ord}$, $\beta < \alpha$ (i.e., $\beta \in \alpha$) $\Rightarrow \beta \not\approx \alpha$. The boxes $\boxed{\dots}$ above are cardinals. $\mathrm{Card} = \{0, 1, 2, \dots, \omega, \omega_1, \dots\} = \{0, 1, 2, \dots; \aleph_0, \aleph_1, \aleph_2, \dots, \aleph_{\omega_1}, \dots\}$ are cardinals (again a proper class), where $\aleph_0 = \omega$, $\aleph_1 = \omega_1$.

- $\forall$ set X $\exists!$ cardinal $\varkappa$ s.t. $X \approx \varkappa =: |X|$, the

④ Cardinality of X. $|X_1 \times X_2 \times \dots \times X_n| = \max_{1 \le i \le n} |X_i|$ if some X_i is infinite.

The Continuum Hypothesis = CH:

$|\underbrace{\{0,1\} \times \{0,1\} \times \dots\dots}_{\aleph_0 (=\omega = \mathbb{N}) \text{ factors}}| = \aleph_1$, i.e. $2^{\aleph_0} = \aleph_1$.

P. Cohen: not true, may be larger: $= \aleph_2, \aleph_{30}, \dots$
1965

$\boxed{\text{Zorn lemma}}$ $(X, \le)$ is s.t. ~~for~~ every $A \subseteq X$ that is linearly ordered by $\le$ has an upper bound. Then $\forall a \in X$ $\exists b \in X: a \le b$ & b is maximal in $(X, \le)$ (there is no larger element in X ~~than~~ than b).

$\Updownarrow$

AC

This is often used in algebra, analysis, ... E.g., in every ring there is a maximal ideal.

Relational systems, homomorphisms (I will write shortly „hom." and „homs".) X, $\Omega \neq \emptyset$ sets

$R \subseteq X^\Omega$ .... an Ω-ary relation on X

$|\Omega| = 1$: $R \subseteq X$ unary rel., $|\Omega| = 2$: $R \subseteq X \times X$: binary rel., ..., Let $R \subseteq X^n$, $S \subseteq Y^n$, $f: X \to Y$, $(X,R) \to (Y,S)$. Then f is ~~a~~ a hom. (w.r.t. the n-ary relations R and S) if $\forall \xi \in R: f \circ \xi \in S$. For example: unary case is that $f[R] \subseteq S$, binary case: $aRb \Rightarrow f(a) S f(b)$, ...

$\boxed{\text{Problem 5.3}}$ We define, in this situation, that f is an isomorphism if $\exists g: (Y,S) \to (X,R)$ a hom. s.t.

$$f \circ g = id_{(Y,S)}$$
$$g \circ f = id_{(X,R)}$$

$\Longleftrightarrow$ f is a bijection from X to Y and $\forall \xi \in X^n: f \circ \xi \in S \Leftrightarrow \xi \in R$. Prove this equivalence.

type: $\Delta = (\Delta_t)_{t \in T}$ $(\Delta_t, T \neq \emptyset)$

- **finite**: all Δ_t and T are finite
- **finitary**: all Δ_t are finite.

A relational ~~system~~ structure ~~on a set~~ of type Δ on a set X $(\neq \emptyset)$ (the ground set) is a system $R = (R_t)_{t \in T}$ with $R_t \subseteq X^{\Delta_t}$ i.e. each R_t is Δ_t-ary relation on X. Then

If $(X,R), (Y,S)$ are rel. structures (of type Δ) then $f: X \to Y$ is a hom. if it is a hom. w.r.t. to R_t, S_t for $\forall t \in T$.