L24, May 18, 2020 — Separation axioms

These regulate how rich the given topology $(X,\mathcal{T})$ is.

$\boxed{(X,\mathcal{T}) \text{ is a } T_0 \text{ space}}$ $\forall x,y\in X,\ x\neq y\ \exists U\in\mathcal{T}:$
$: x\in U\not\ni y$ or $y\in U\not\ni x$.

$\boxed{(X,\mathcal{T}) \text{ is a } T_1 \text{ space}}$ $\forall x,y\in X,\ x\neq y\ \exists U\in\mathcal{T}:$
$: x\in U\not\ni y$

$(X,\mathcal{T})$ is $T_1 \iff \forall$ finite set is closed.

$(X,\mathcal{T})$ is T_2 space or Hausdorff space: $\forall x,y\in X,\ x\neq y\ \exists U,V\in\mathcal{T}:$
$: x\in U,\ y\in V,\ U\cap V=\emptyset$

these spaces are best known.

It is not hard to see:

Thm. If $f,g: X\to Y$ are continuous maps and Y is H. space. Then $\{x \mid f(x)=g(x)\}\subseteq X$ is a closed set.

~~Let~~ ~~be~~ $(X,\mathcal{T})$ a topology, we say that $M\subseteq X$ is dense (in ~~X~~) if $\overline{M}=X$. From the thm. we have:

Corollary Let $f,g: X\to Y$ ~~are~~ be continuous and Y be H. space and $f|_M=g|_M$ for some dense set $M\subseteq X$.

Then $f = g$.

(X,T) is a *regular space* or a **T_3 space**: $\forall x \in X$ $\forall$ closed set $A \subseteq X$, $x \notin A$ $\exists U, V \in T$: $x \in U$, $A \subseteq V$, $U \cap V = \emptyset$.

Regular spaces are characterized (in the LN of prof. A. Pultr) by Thm. 5.4.1.

(X,T) is a *completely regular space* or a **$T_{3\frac{1}{2}}$ space**: $\forall x \in X$ $\forall$ closed set $A \subseteq X$, $x \notin A$, $\exists$ continuous map $f: X \to [0,1]$ s.t. $f(x) = 0$ and $f|A \equiv 1$. Here $[0,1]$ is the unit real interval with the Euclidean topology.

See Thm. 5.5.2 in the LN for a characterization of $T_{3\frac{1}{2}}$ spaces.

(X,T) is a *normal space* or a **T_4 space**

~~$\forall$ closed sets $A, B \subseteq X$, $A \cap B = \emptyset$ $\exists$ continuous map $f: X \to [0,1]$ s.t. $f|A \equiv 0$ and $f|B \equiv 1$~~

$\forall$ closed sets $A, B \subseteq X$, $A \cap B = \emptyset$ $\exists U, V \in T$: $A \subseteq U$, $B \subseteq V$, $U \cap V = \emptyset$.

$\Longleftrightarrow T_4$.

Separation by a continuous function, which was here, is in fact

$\boxed{\text{theorem (Urysohn's lemma)}}$ (X,τ) is normal $\Leftrightarrow$ $\Leftrightarrow$ $\forall$ closed sets $A, B \subseteq X$, $A \cap B = \emptyset$ $\exists$ continuous map $f: X \to [0,1]$ s.t. $f|A \equiv 0$ & $f|B \equiv 1$. Proof: – see the LN; $\Leftarrow$ is easy: $U := f^{-1}([0,\frac{1}{2}))$, $V := f^{-1}((\frac{1}{2},1])$

I already mentioned in the previous lecture (of mine) that every metrisable space is Hausdorff. But it is in fact normal.

$\boxed{\text{Proposition}}$ Any metric space (M,ρ) is normal as a topological space. Proof. For $X \subseteq M$ and $a \in M$ we define the distance between a and X by

$$\rho(a,X) := \inf_{b\in X} \rho(a,b).$$

For two disjoint closed sets $A, B \subseteq M$ we then take the function $f: X \to [0,1]$, $f(x) := \dfrac{\rho(x,A)}{\rho(x,A)+\rho(x,B)}$. One can show (do it as an exercise) that the denominator is always $\neq 0$, that f is continuous and that f goes to $[0,1]$. From the def. it is obvious that $x \in A \Rightarrow f(x) = 0$ and $x \in B \Rightarrow f(x) = 1$. Thus we ha-

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ve a separating function as in the easy $\Leftarrow$ in Urysohn's lemma and (M, S) is normal in the topological sense. $\boxtimes$

We have the hierarchy of separation axioms:

$$T_0 \Leftarrow T_1 \Leftarrow T_2 \Leftarrow T_3 \,\&\, T_1 \Leftarrow T_{3\frac{1}{2}} \,\&\, T_1 \Leftarrow T_4 \,\&\, T_1.$$

None of these implications can be reversed, but for examples are hard (so there is no U.-lemma for regular spaces).

By Thm. 5.8.1, properties T_i, $i = 0, 1, 2, 3$ and $3\frac{1}{2}$, are preserved under taking a subspace and under products. Normality is not preserved under these operations.

By the following theorem, $T_{3\frac{1}{2}} \,\&\, T_1$ spaces are close to be metrisable, each such space is (up to homeomorphism) a subspace of a possibly infinite-dimensional cube.

$\boxed{\text{Theorem (Tichonov's embedding theorem)}}$

$(X, \mathcal{T})$ is $T_{3\frac{1}{2}} \,\&\, T_1 \iff (X, \mathcal{T})$ is homeomorphic to a subspace of the cube $[0,1]^M$ for some set M.

Here $[0,1]^M = \prod_{m \in M} X_m$ where $X_m = [0,1]$ for $\forall m$.

Proof: See the LN.

Compact spaces

Compactness is perhaps the most useful topological property.

An (open) cover of a top. space $(X, \mathcal{T})$ is a subset $\mathcal{U} \subseteq \mathcal{T}$ s.t. $\bigcup \mathcal{U} = X$. A cover of a set $Y \subseteq X$ is a $\mathcal{U} \subseteq \mathcal{T}$ satisfying $\bigcup \mathcal{U} \supseteq Y$.

$\boxed{(X, \mathcal{T}) \text{ is compact}}$ "Every cover has a finite subcover." Formally, $\forall \mathcal{U} \subseteq \mathcal{T}$ s.t. $\bigcup \mathcal{U} = X$ $\exists \mathcal{V} \subseteq \mathcal{U}$, s.t. $\mathcal{V}$ is finite and still $\bigcup \mathcal{V} = X$.

A subset $A \subseteq X$ is compact if the subspace $(A, \mathcal{T}|A)$ is compact, i.e. $\forall \mathcal{U} \subseteq \mathcal{T}$ s.t. $\bigcup \mathcal{U} \supseteq A$ $\exists \mathcal{V} \subseteq \mathcal{U}$ s.t. $\mathcal{V}$ is finite and $\bigcup \mathcal{V} \supseteq A$.

For the last lecture by prof. A. Pultr on May 19, please study the rest of Chapter <u>V.6 Compactness</u>, with the exclusion of V.6.8 (Čech-Stone compactification), in the LN of prof. A. Pultr.